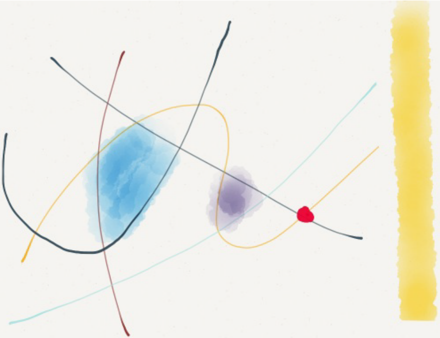
Calabi-Yau Geometry and Mirror Symmetry Conference



$$\begin{pmatrix} x & z & y \\ z & y & x \\ y & x & z \end{pmatrix} \bullet \begin{pmatrix} x^2 - yz & z^2 - xy & y^2 - zx \\ z^2 - xy & y^2 - zx & x^2 - yz \\ y^2 - zx & x^2 - yz & z^2 - xy \end{pmatrix}$$

$$= (x^3 + y^3 + z^3 - 3xyz) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Homological mirror functor from counting polygons

Cheol-Hyun Cho (Seoul National Univ.)
(based on a joint work with Hansol Hong and Siu-Cheong Lau)

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and another hypersurface in $\mathbb{P}^4/(\mathbb{Z}/5)^3$

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- Holomorphic curve counting invariants of X equals certain period integrals of Y.

Mirror Symmetry between a space and a function

- Another version of mirror symmetry relates **space** X, with a **potential function** $W: Y \to \mathbb{C}$ on Y.
- Simplest example is the case of sphere.

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$$W(z) = z + \frac{q}{z}.$$

- Quantum cohomology of $\mathbb{C}P^1$ is isomorphic to the Jacobian ring of W; $\mathbb{C}[z, z^{-1}]/\partial W$.

$$\partial W = 1 - \frac{q}{z^2}$$

  gives $z^2 = q$, which corresponds to $[pt] *_Q [pt] = q[S^2]$.
- This example generalizes to all toric manifolds.

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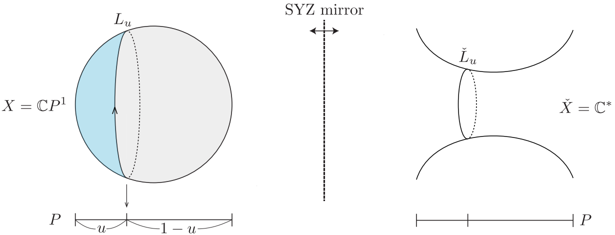

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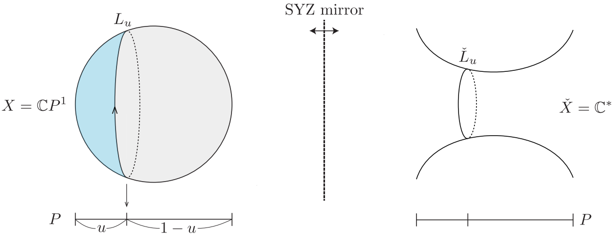

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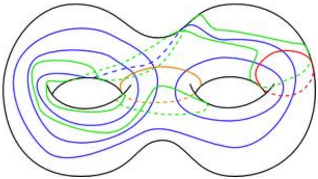
• It becomes more difficult to use when fibers get more singular.

Fukaya category of a surface Σ



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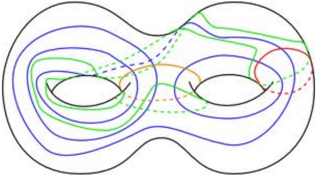

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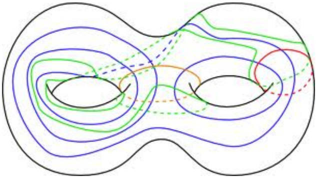

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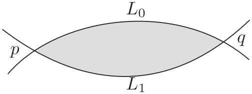
  $Hom(L_i, L_i)$ is given by a Morse complex of $L_i \cong S^1$
- There are operations on Morphisms:
  differential m_1, product m_2, homotopy for associativity m_3, $\cdots$, which defines an A_∞-category.

Operations on Morphisms

- (Differential m_1)

$$m_1 : Hom(L_0, L_1) \to Hom(L_0, L_1)$$

counts immersed **bi-gons** with boundaries on L_0 and L_1.



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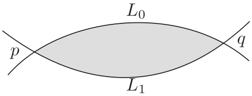

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- We use Novikov ring

$$\Lambda = \{\sum a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lim_{i \to \infty} \lambda_i = \infty\}$$

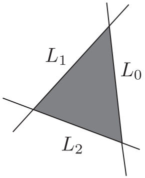
- Corners of the bigon should be locally convex

Operations on Morphisms 2

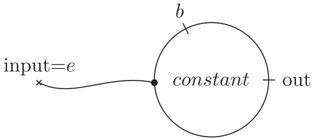
- (Product m_2)

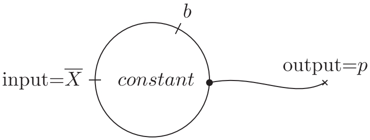
$$m_2 : Hom(L_0, L_1) \times Hom(L_1, L_2) \to Hom(L_0, L_2)$$

counts **triangles** with boundaries on L_0 and L_1 and L_2.



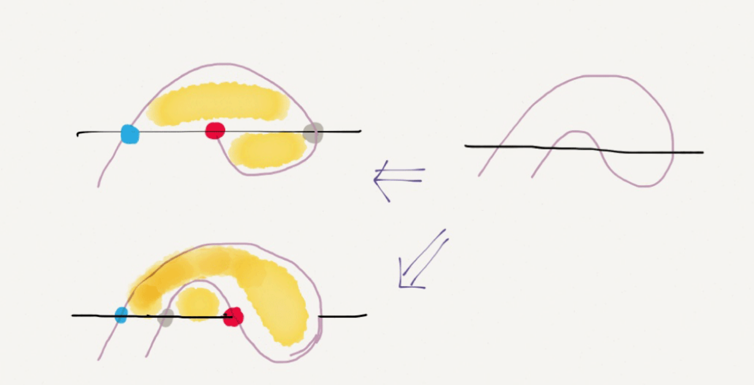
- In general (if $L_i = L_j$), we also count with additional Morse trajectory contributions.





Differential $m_1^2 = 0$

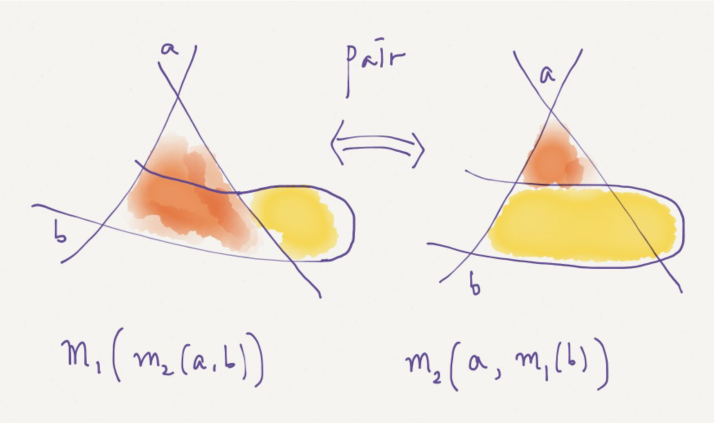
- $m_1^2 = 0$ can be proved, since there are two matching diagrams.



Operations on Morphisms

- These satisfy product rule (up to $\pm$):

$$m_1(m_2(a,b)) = m_2(m_1(a),b) + m_2(a, m_1(b))$$



Operations on Morphisms

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- In general, we can define m_k counting k-**gons** with areas.
- $m_1, m_2, m_3, \cdots$ satisfy A_∞-equations. Given n-inputs $x_1, \cdots, x_n$,

$$\sum_{k_1+k_2=n+1} m_{k_1}(x_1, \cdots, m_{k_2}(\cdots), \cdots, x_n) = 0$$

- **Fukaya category** can be defined in any symplectic manifold:
  • Objects are Lagrangian submanifolds, and Morphisms are their intersection points, and operations by counting J-holomorphic polygons.

Matrix Factorization Category

- Given a (Laurant) polynomial W, for example,

$$\begin{aligned} W(z) &= z^n \\ W(x,y,z) &= x^3 + y^3 + z^3 - 3xyz \\ W(z) &= z + \frac{q}{z} \end{aligned}$$

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- For $W = z + \frac{q}{z}$, it has a critical value $\lambda = 2\sqrt{q}$ and

$$W - \lambda = (z - \sqrt{q}) \bullet (1 - \frac{\sqrt{q}}{z})$$

Matrix Factorizations

- We can factor W into two matrices as follows.
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- We have matrices M_0, M_1 such that

$$M_0 \bullet M_1 = W \cdot Id, \qquad M_1 \bullet M_0 = W \cdot Id.$$

Category of Matrix Factorizations

- Let $R = \mathbb{C}[x, y, z]$ polynomial ring. Write $R \oplus R \oplus R = R^{\oplus 3}$
- Consider the matrices as maps

$$R^{\oplus 3} \xrightarrow{M_0} R^{\oplus 3} \xrightarrow{M_1} R^{\oplus 3}$$

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- **Matrix factorization of** W is two R-modules P_0, P_1 with maps

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Category of Matrix Factorizations 2

- Given two matrix factorizations, we define morphisms between them

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$$\Downarrow f \qquad \Downarrow f \qquad \Downarrow f$$

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- Differential m_1 on Morphisms by

$$N_i \circ f - f \circ M_i$$

- m_2 is the composition of maps.
- This defines **differential graded category**, which is an A_∞-category with $m_{\geq 3} = 0$.

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- $MF(W) := \amalg_{\lambda \in Crit(W)} MF(W, \lambda)$
- It is also called the category of singularity of W.

Homological mirror functor

In this setting, Homological mirror symmetry conjectures:

• There exist a derived equivalence between the geometric A_∞-category to the algebraic dg-category:

$$\text{Fukaya Category of } X \quad \leftrightarrow \quad \text{Matrix Fact.}(W)$$

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• There exist an A_∞ functor relating A_∞-operations on X and dg-operations on $MF(W)$.

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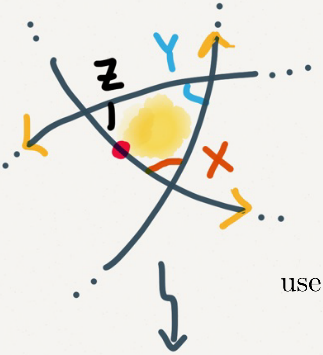
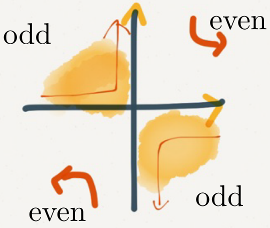
   $$\mathcal{LM}^L : \mathcal{F}uk(X) \longrightarrow Mat.Fact.(W_L)$$
   constructed from geometric data (via Fukaya category operations)
3. **If** $\mathcal{LM}^L$ is an equivalence, this would prove homological mirror symmetry (**Generalized SYZ**).

Potential W_L from Lagrangian Immersion L



$xyz\ q$



use only odd intersections as inputs

Define W by counting Polygons recording corners, and areas passing through a generic point

W_L as Lagrangian Floer Potential

- Given a Lagrangian immersion L, consider its A_∞-algebra, and degree one immersed generators $X_1, \cdots, X_n$.

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- Suppose that

$$b = x_1 X_1 + x_2 X_2 + \cdots + x_n X_n$$

becomes weak bounding cochains, or

$$m_1(b) + m_2(b, b) + m_3(b, b, b) + \cdots = W(b) \cdot e$$

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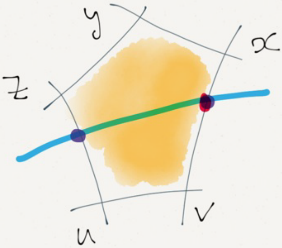
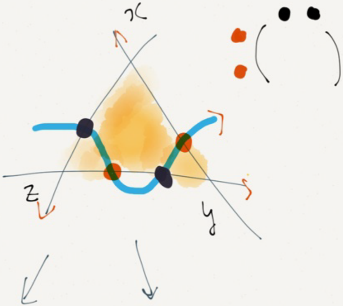
is a multiple of unit.
- Then, the coefficient $W(b)$ is a function in $\Lambda[[x_1, \cdots, x_n]]$, defines localized mirror of L.

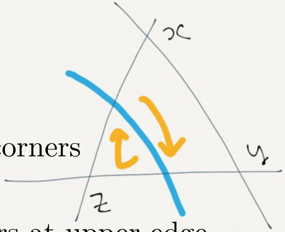
Matrix Factorization corresponding to L_1

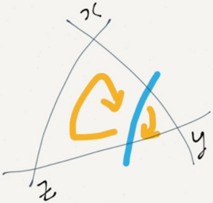


$$xyzuv = xyz \cdot uv$$



count bigons with corners





with possible corners at upper edge

Matrix Factorization corresponding to L_1

In general, consider L_1 transversely intersecting the immersed Lagrangian L.

- Intersection $L_1 \cap L$ consists of even and odd intersection points.
- We count bigons allowing odd degree turns at the upper boundary.

$$m_1^{b,0}(p) = \sum_{k=0}^{\infty} m_{k+1}(b, b, b, b, b, \cdots, b, p)$$

- counting the bigon gives an element $\Lambda_0[x_1, \cdots, x_n]$.
- This deformed complex

$$(CF((L, b), L_1), m_1^{b,0})$$

satisfies $(m_1^{b,0})^2 = W(b)$, providing the matrix factorization of $W(b)$.

Localized mirror functor

We define an A_∞-functor

$$\phi_L : \mathcal{F}uk(X) \to MF(W_L)$$

Say ϕ_L sends Lagrangian L_j to Matrix factorizations M_j.

$$(\phi_L)_1 : CF(L_i, L_j) \to Mor(M_i, M_j)$$

$$(\phi_L)_1(p_{ij})(\cdot) = \pm m_2^{b,0,0}(\cdot, p_{ij})$$

In general we define

$$(\phi_L)_k : CF(L_1, L_2) \otimes \cdots \otimes CF(L_{k-1}, L_k) \to Mor(M_1, M_k)$$

$$(\phi_L)_k(p_{12}, p_{23}, \cdots, p_{(k-1)k})(\cdot) = \pm m_{k+1}^{b,0,\cdots,0}(\cdot, p_{12}, p_{23}, \cdots, p_{(k-1)k})$$

Example of orbi-spheres

Cubic mirror symmetry:

$$x^3 + y^3 + z^3 = 0 \quad \text{in} \quad \mathbb{P}^2$$

$$\longleftrightarrow x^3 + y^3 + z^3 - \sigma xyz = 0 \quad \text{in} \quad \mathbb{P}^2/(\mathbb{Z}/3)$$

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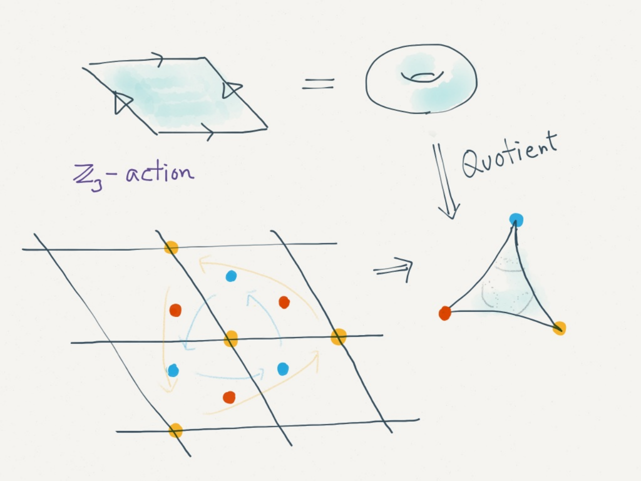
1. Zero set of $x^3 + y^3 + z^3 = 0$ in $\mathbb{P}^2$ gives the elliptic curve E (three copy of E).
2. The relationship between the Kähler parameter q and complex parameter σ was explained by Siu-Cheong Lau.
3. Further quotient by $\mathbb{Z}/3$-action, gives an orbi-sphere $\mathbb{P}^1_{3,3,3}$.

Quotient orbi-sphere $\mathbb{P}^1_{3,3,3}$

We take a $\mathbb{Z}_3$ quotient of the elliptic curve



Main example

- Our first goal is to define mirror potential W of the form

$$W(x,y,z) = x^3 + y^3 + z^3 - \sigma xyz$$

so that we have mirror symmetry

$$\text{Eliptic curve} \longleftrightarrow (W : (\mathbb{C}^3/\mathbb{Z}_3) \to \mathbb{C})$$

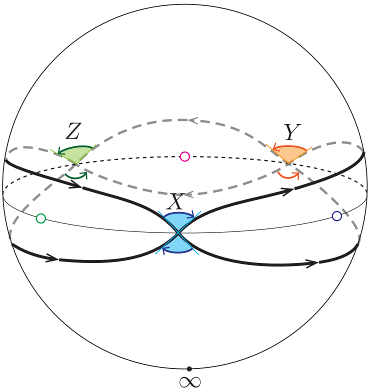
or

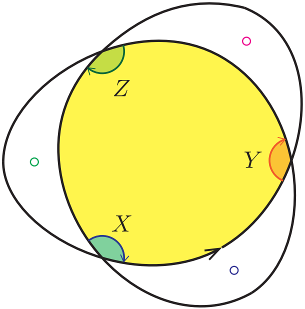
$$\text{Eliptic curve}/\mathbb{Z}_3 \longleftrightarrow (W : \mathbb{C}^3 \to \mathbb{C})$$

- In general, we consider $X = \mathbb{P}^1_{a,b,c}$ and prove homological mirror symmetry between $X = \mathbb{P}^1_{a,b,c}$ and $W = x^a + y^b + z^c - \sigma xyz + \cdots$ when $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$, by constructing a mirror functor.

Immersed Lagrangian

Our main object is the following immersed curve in the orbi-sphere (Seidel considered such Lagrangian immersion to prove HMS for genus two curve)





Weakly unobstructed

Proposition

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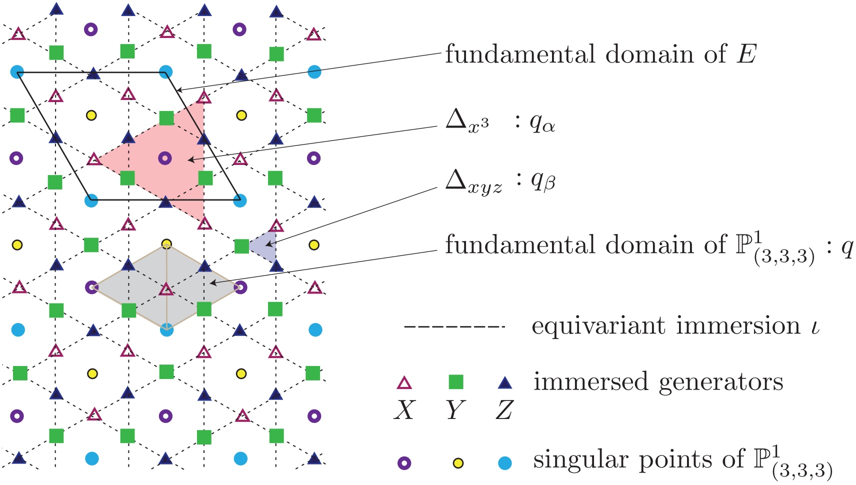
1. Reflection preserves the orientation of the Lagrangian, but the complex orientation is the opposite, which gives rise to $(-1)^k$
2. The number of times P or P^{op} passes through generic point (for non-standard spin structure) is $k+1$ mod 2. (If P and P^{op} covers minimal edges $6l$ times, then P covers, $3l$ times. But each edge of P consists of odd number of minimal edges. Hence $l = k+1$ mod 2.)

Immersed Lagrangian Upstairs

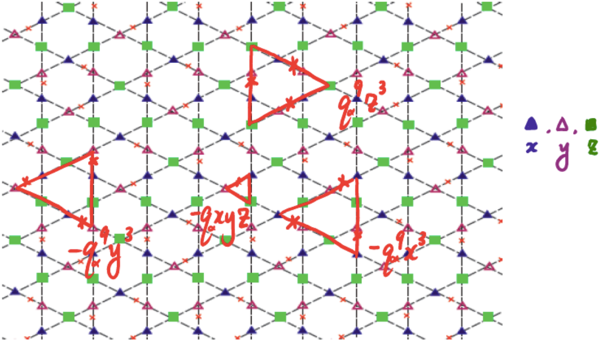
Immersed curve are straight lines in the universal cover.



A miracle

Generating function: $W \doteq \sum_{\beta} n_\beta q^\beta z^{\partial\beta}$

$$= q_\alpha xyz - q_\alpha^9 x^3 - q_\alpha^9 y^3 + q_\alpha^9 z^3 + \ldots\ldots$$



Construction of mirror potential

$$\begin{aligned} W &= qxyz - q^9x^3 - q^9y^3 + q^9z^9 + \cdots \\ &= -\phi(q)(x^3 + y^3 - z^3) + \psi(q)xyz. \end{aligned}$$

where

$$\begin{aligned} \phi &= \sum_{k=0}^{\infty}(-1)^{3k+1}(2k+1)q_\alpha^{3(12k^2+12k+3)} \\ \psi &= -q_\alpha + \sum_{k=1}^{\infty}(-1)^{3k}\left(-(6k+1)q_\alpha^{(6k+1)^2} + (6k-1)q_\alpha^{(6k-1)^2}\right). \end{aligned}$$

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After change of coordinate, we have

$$x^3 + y^3 + z^3 - \frac{\psi}{\phi}xyz$$

General a, b, c case

- In general, one can consider $\mathbb{P}^1_{a,b,c}$, where each orbifold point has $\mathbb{Z}_a, \mathbb{Z}_b, \mathbb{Z}_c$ singularity. We have

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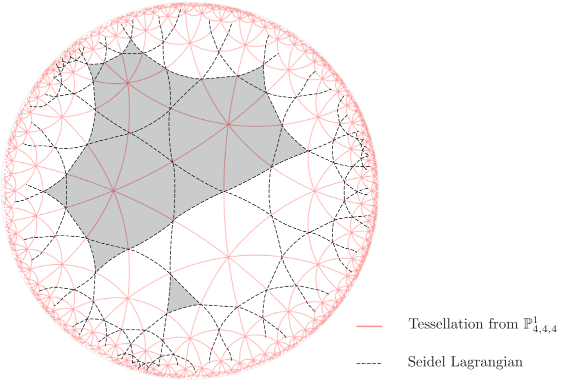
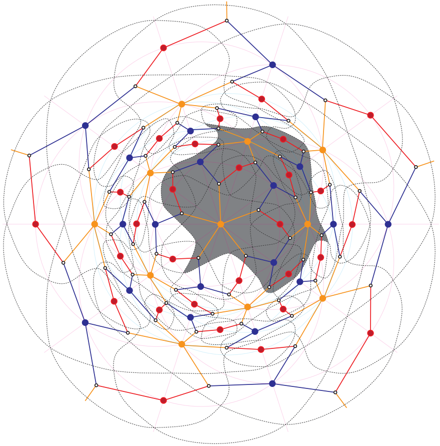
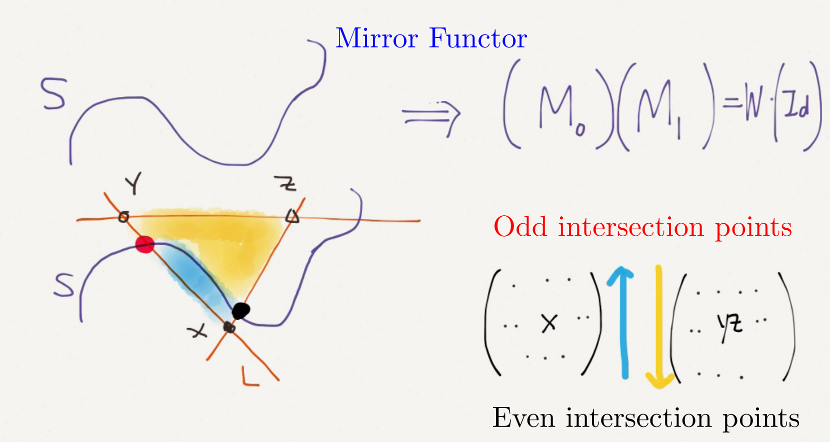
- We can count all necessary polygons to compute W explicitly (joint work with Sang-hyun Kim)



- The case of $(2, 3, 5)$

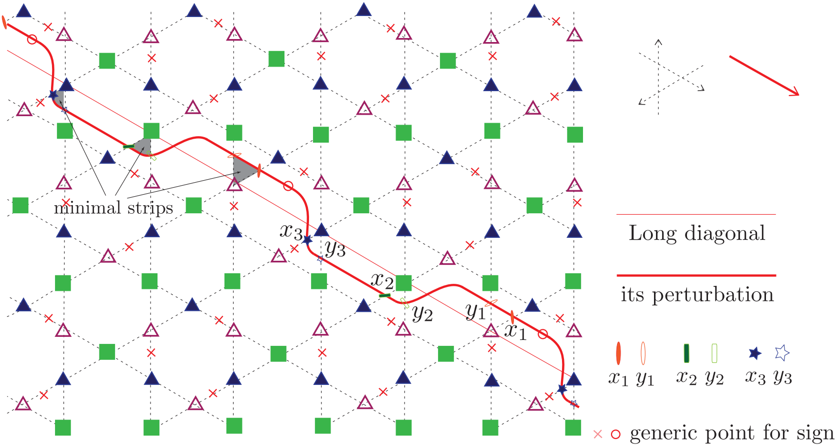


Homological Mirror functor in action



Equator $\leftrightarrow$ (3×3) matrix factorization of W

- Equator (straight line passing through 3 orbifold points upstairs) corresponds to (3×3) matrix factorization of W explained before.



Mirror matrix factorization

$$J = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} \beta_1 x^2 + \gamma_1 yz & \beta_3 z^2 + \gamma_3 xy & \beta_2 y^2 + \gamma_2 zx \\ \beta_2 z^2 + \gamma_2 xy & \beta_1 y^2 + \gamma_1 zx & \beta_3 x^2 + \gamma_3 yz \\ \beta_3 y^2 + \gamma_3 zx & \beta_2 x^2 + \gamma_2 yz & \beta_1 z^2 + \gamma_1 xy \end{pmatrix} \end{array}$$

$$E = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} \alpha_1 x & \alpha_2 z & \alpha_3 y \\ \alpha_3 z & \alpha_1 y & \alpha_2 x \\ \alpha_2 y & \alpha_3 x & \alpha_1 z \end{pmatrix} \end{array}.$$

This satisfies

$$J \bullet E = E \bullet J = W \cdot Id$$

Matrix factorization of wedge-contraction type

- Consider $W \in R = \mathbb{C}[x_1, \cdots, x_n]$ with isolated singularity at the origin.
- Consider **odd variables** $\theta_1, \cdots, \theta_n$ so that we have

$$\theta_i \cdot \theta_j = -\theta_j \cdot \theta_i, \quad \partial_{\theta_i}\partial_{\theta_j} = -\partial_{\theta_j}\partial_{\theta_i}$$

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- Define

$$\delta = \sum_i x_i \theta_i + w_i \partial_{\theta_i}.$$

$$\delta^2 = W$$

Matrix factorization of wedge-contraction type II

- $(R < \theta_1, \cdots, \theta_n >, \delta)$ defines a matrix factorization of W.
- If $n = 3$, then $2^3 = 8$, and even and odd parts of $R < \theta_1, \cdots, \theta_n >$ is 4 dimensional. Hence provides (4×4) matrix factorization of W.

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- Dyckerhoff proved that this matrix factorization generates the Matrix factorization category if W has only isolated singularity at the origin.
- This matrix factorization corresponds to the skyscraper sheaf at the singular point.

Seidel Lagrangian maps to wedge-contraction matrix factorization

- To prove homological mirror symmetry, we prove that Seidel Lagrangian L maps to (4×4) matrix factorization.
- First $CF(L, L)$ has 8 generators.

$$(min, max, X, \bar{X}, Y, \bar{Y}, Z, \bar{Z})$$

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- After this quantum corrections, this become wedge-contraction type.
- We show that $W_{a,b,c}$ has isolated singularities at the origin for $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$. (joint with S.H. Kim) In such case, Seidel Lagrangian also split-generates derived Fukaya category of $\mathbb{P}_{a,b,c}$ (using Abouzaid, AFOOO),
- This proves the homological mirror symmetry.

Equivariant theory

- In elliptic curve quotient $E/\mathbb{Z}_3$, Lagrangian is given by $\mathbb{Z}_3$-equivariant brane in E.
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- We get upstairs functor to equivariant Matrix factorization category.

$$\mathcal{F}uk(E) \to \mathcal{MF}_{\check{G}}(W)$$