International Journal of Mathematics
Vol. 26, No. 1 (2015) 1550001 (28 pages)

DOI: 10.1142/S0129167X15500019

On weighted $W^{2,p}$ estimates for elliptic equations with BMO coefficients in nondivergence form

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Received 5 January 2014
Accepted 9 December 2014
Published 8 January 2015

We establish global weighted $W^{2,p}, 2 < p < \infty$, estimates for the solution to the Dirichlet problem for an elliptic equation in nondivergence form with BMO coefficients in a $C^{1,1}$ domain under the assumption that the matrix of the coefficients has a small BMO seminorm while the associated weight belongs to a Muckenhoupt class. These conditions are weaker than those reported in the literature.

Keywords: Elliptic equation; weighted L^p space; BMO space; strong solution; Muckenhoupt weight.

Mathematics Subject Classification 2010: 35J15, 46E30, 46E35

1. Introduction

In this paper, we consider the following Dirichlet problem:

$$\begin{cases} a_{ij} D_{ij} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and the matrix $\mathbf{A} = (a_{ij})$ of coefficients is assumed to be symmetric and uniformly elliptic, see (2.1)–(2.2).

We are here concerned with optimal weighted L^p regularity results for solutions to (1.1). More precisely, our goal is to find minimal conditions both on the coefficients a_{ij} and on the boundary $\partial\Omega$ of the domain under which we derive the following global weighted $W^{2,p}$ estimate

$$\|D^2u\|_{L^p_w(\Omega)} \leq c\|f\|_{L^p_w(\Omega)}, \quad \forall\, p \in (2,\infty) \tag{1.2}$$

with a weight w belonging to the Muckenhoupt class $A_{\frac{p}{2}}$, where the constant $c > 0$ is independent of f and u.

For the unweighted case that $w = 1$ in (1.2), it is well known that there does not exist a unique strong solution in $W^{2,p}(\Omega)$ to (1.1) under the basic structural conditions on the coefficients like (2.1)–(2.2), even if the domain has an appropriate smoothness condition, as we see from [26, 29, 31]. It also turned out that this classical Dirichlet problem could not be solvable in an arbitrary bounded domain in $\mathbb{R}^n$ due to the famous examples of Zaremba and Lebesgue in [24, 37]. These facts naturally lead us to impose both a suitable additional condition on the coefficients and a certain geometric restriction on the boundary of the domain, in order to achieve the unique solvability of (1.1) in $W^{2,p}(\Omega)$ for the full range of $p \in (1,\infty)$.

As the classical results, if the coefficients a_{ij} are continuous and the boundary $\partial\Omega$ of the domain Ω belongs to C^2, then the estimate (1.2) of the problem (1.1) is valid for every $1 < p < \infty$, see [18, 26].

In the case of discontinuous coefficients, Miranda in [27] proved the well-posedness of (1.1) in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ when the coefficients a_{ij} belong to $W^{1,n}$ and $\partial\Omega$ is sufficiently regular. This is known for an optimal result among the L^p spaces for the nonuniqueness of the solutions in the case that $a_{ij} \in W^{1,n-\epsilon}, \epsilon > 0$. The assumptions $W^{1,n}$ in [27] were relaxed in [3] to the level that the first derivatives of the coefficients a_{ij} are $L(n,\infty)$, which is the so-called Marcinkiewicz space. Since then, there have been further research activities on the $W^{2,p}$ regularity problem, and especially, in the papers [11, 12], Chiarenza, Frasca and Longo proved the interior and boundary $W^{2,p}$ estimates of solutions to (1.1) when the coefficients a_{ij} belong to VMO and $\partial\Omega$ belongs to $C^{1,1}$. The approach in [11, 12] was mainly based on the explicit representation formulas involving singular integral operators and commutators. This approach was later generalized and applied by Palagachev, Di Fazio, Maugeri and Softova to the quasilinear elliptic problems, see [14, 25, 26, 30]. In [22], Krylov proposed a different approach for the $W^{2,p}$ solvability of solutions to the nondivergence type equations with VMO coefficients, which was mainly based upon the use of pointwise estimates of the sharp function of second-order derivatives of solutions. Many studies on L^p regularity have been done via this approach as, for instance, in [15, 16, 21, 34]. There is another approach, the so-called maximal function free technique, which was introduced by Acerbi and Mingione [1] and employed later in [36] to obtain the Orlicz regularity for second-order elliptic equations in nondivergence form with small BMO coefficients. This approach, not using either representation formulas or maximal functions, is suitable to the cases that a scaling in time and space is given differently such as p-Laplacian parabolic equations and systems, see, for instance, [4, 8, 17, 23].

In accordance with such research achievements on the L^p regularity, we focus on establishing the global weighted $W^{2,p}$ estimates for the Dirichlet problem (1.1), in particular, when coefficients a_{ij} have small BMO semi-norms and the domain Ω is a bounded subset of $\mathbb{R}^n$ and its boundary $\partial\Omega$ belongs to $C^{1,1}$. Indeed, our results in this paper can be considered as a natural extension of those in [12]. To be more exact, the L^p regularity of (1.1) in [12] is a special case of the weighted L^p regularity of (1.1) when a weight $w = 1$. Moreover, it is worth mentioning that the class of the coefficients which we are treating in this paper, strictly contains VMO and so $W^{1,n}$, which were previously considered, for instance, in the works [5, 12, 22, 27].

Our approach is strongly influenced by [6, 7, 10, 33, 35]. Unlike the approaches in [1, 12, 22], we use the Hardy–Littlewood maximal function as the basic tool, to deduce the required power decay estimates for the weighted measure of the upper level sets for the maximal function of the second derivatives of the solutions. In particular, an essential part in our approach is to find a local estimate of solutions of the problem (1.1) by comparison with those of the limiting problems with constant coefficients of the local average values of the coefficients of (1.1). Furthermore, a weighted covering lemma and the standard flattening argument contribute largely to derive the required global weighted $W^{2,p}$ estimate along with interior and boundary weighted $W^{2,p}$ estimates.

This paper is organized as follows. In the next section we present the relevant notations, definitions and auxiliary lemmas to state the main results. In Sec. 3 we establish the interior weighted $W^{2,p}$ estimates, and then obtain the weighted $W^{2,p}$ estimates near flat boundary in Sec. 4. By means of covering and flattening arguments, we finally derive the global weighted $W^{2,p}$ estimate in Sec. 5.

2. Preliminary Tools and Main Result

We start this section with standard notations and definitions, and recall some lemmas which are basic tools to obtain the main results. Let $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $B_r^+(x) = B_r(x) \cap \{x_n > 0\}$. For the sake of simplicity, we write $B_r = B_r(0)$ and $B_r^+ = B_r^+(0)$. We also denote $T_r(x) = B_r(x) \cap \{x_n = 0\}$ and $T_r = B_r \cap \{x_n = 0\}$.

The following definitions are associated with the conditions of the coefficient matrix $\mathbf{A} = (a_{ij})$. We first assume that $\mathbf{A} = (a_{ij}) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a measurable and matrix-valued function on $\mathbb{R}^n$ with the symmetric condition $a_{ij} = a_{ji}$.

Definition 2.1. We say that the coefficient matrix $\mathbf{A}$ is *uniformly elliptic* if there is a positive constant Λ such that

$$\Lambda^{-1}|\xi|^2 \le \langle \mathbf{A}(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \quad \text{for } \forall\, \xi \in \mathbb{R}^n \text{ and a.e. } x \in \mathbb{R}^n. \tag{2.1}$$

Definition 2.2. We say that the coefficient matrix $\mathbf{A} = (a_{ij})$ is (δ, R)-*vanishing* if

$$\sup_{0<r\le R}\sup_{x\in\mathbb{R}^n}\left(\fint_{B_r(x)} |\mathbf{A}(y) - \overline{\mathbf{A}}_{B_r(x)}|^2\, dy\right)^{\frac{1}{2}} \le \delta, \tag{2.2}$$

where $\overline{\mathbf{A}}_{B_r(x)} = \frac{1}{|B_r(x)|}\int_{B_r(x)} \mathbf{A}(y)dy$ is the average value of $\mathbf{A}$ on the ball $B_r(x)$.

In the above definition, R can be any positive number by scaling the given equations, whereas δ is invariant under such scaling. A locally integrable function f is called *of bounded mean oscillation on* $\mathbb{R}^n$, denoted by $f \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_* := \sup_{B\subset\mathbb{R}^n} \fint_B |f - \overline{f}_B|\, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. In the whole paper, we assume that $\mathbf{A} = (a_{ij})$ is in the John–Nirenberg space BMO of functions of bounded mean oscillation with small BMO semi-norms, which we defined above in (2.2). This is a more general concept than the VMO condition appeared in other papers such as [12, 22]. Since the coefficients a_{ij} can be extended in $\mathbb{R}^n$ preserving the small BMO condition (see [2]), we can consider the small BMO coefficients a_{ij} to be defined in $\mathbb{R}^n$ throughout this paper. Moreover, we notice that the condition (2.2) is equivalent to the small BMO condition $\|\mathbf{A}\|_* \le \delta$ by the John–Nirenberg inequality (see [20] for details).

Before stating our main result, let us present some properties of the Muckenhoupt classes A_s, $1 < s < \infty$, which will be treated in this paper. We say that w is a *weight* in *Muckenhoupt class* A_s, or an A_s *weight*, if w is a positive locally integrable function on $\mathbb{R}^n$ such that

$$[w]_s := \sup\left(\fint_B w(x)dx\right)\left(\fint_B w(x)^{\frac{-1}{s-1}}dx\right)^{s-1} < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. If w is an A_s weight, we write $w \in A_s$, and $[w]_s$ is called the A_s *constant of* w. The A_s class is stable with respect to translation, dilation and multiplication by a positive scalar. Every A_s weight has the doubling property, and the monotonicity $A_{s_1} \subset A_{s_2}, 1 < s_1 \le s_2 < \infty$. A typical example of A_s weights for $1 < s < \infty$ is the function $w_\alpha(x) = |x|^\alpha, x \in \mathbb{R}^n$ where $-n < \alpha < n(s-1)$. We shall identify the weight w with the measure

$$w(E) = \int_E w dx,$$

for measurable sets $E \subset \mathbb{R}^n$.

Related to the A_s weight w is the *weighted Lebesgue space* $L^s_w(\Omega), 1 < s < \infty$, which contains all measurable functions g on Ω such that

$$\|g\|_{L^s_w(\Omega)} := \left(\int_\Omega |g|^s w dx\right)^{1/s} < +\infty.$$

Given $w \in A_s, 1 < s < \infty$ and a non-negative integer m, we also define the *weighted Sobolev space* $W^{m,s}_w(\Omega)$ as the set of functions $g \in L^s_w(\Omega)$ with weak derivatives $D^\alpha g \in L^s_w(\Omega)$ for $|\alpha| \le m$. The norm of g in $W^{m,s}_w(\Omega)$ is given by

$$\|g\|_{W^{m,s}_w(\Omega)} := \left(\sum_{|\alpha|\le m}\int_\Omega |D^\alpha g|^s w dx\right)^{\frac{1}{s}}.$$

The following is an important property of the A_s weights (see [32] for details).

Lemma 2.3. *Let w be an A_s weight for some $1 < s < \infty$, and let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then there exist two constants $\beta, \nu > 0$ depending only on n and w such that*

$$[w]_s^{-1} \left(\frac{|E|}{|B|} \right)^s \leq \frac{w(E)}{w(B)} \leq \beta \left(\frac{|E|}{|B|} \right)^\nu .$$

Unless otherwise stated, we assume that w is an $A_{\frac{p}{2}}$ weight for $2 < p < \infty$ throughout the paper. Let us now state the main theorem in this paper.

Theorem 2.4 (Main Theorem). *Given $2 < p < \infty$ and a weight $w \in A_{\frac{p}{2}}$, there exists a small $\delta = \delta(\Lambda, p, n, w, \partial\Omega) > 0$ so that if $\mathbf{A}$ is uniformly elliptic and (δ, R)-vanishing, $\partial\Omega \in C^{1,1}$ and $|f|^2 \in L_w^{\frac{p}{2}}(\Omega)$, then the solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of (1.1) satisfies $|D^2u|^2 \in L_w^{\frac{p}{2}}(\Omega)$ with the estimate*

$$\int_\Omega |D^2u|^p w dx \leq c \int_\Omega |f|^p w dx,$$

where a constant $c > 0$ is independent of u and f.

A strong solution of Eq. (1.1), which is treated throughout the paper, is a twice weakly differentiable function satisfying Eq. (1.1) almost everywhere in Ω and assuming boundary values on $\partial\Omega$ in classical or in general sense, while a classical solution of the equation must be at least twice continuously differentiable. Since $L_w^{\frac{p}{2}}(\Omega) \subset L^1(\Omega)$ for $2 < p < \infty$, we remark that there is a unique strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of the problem (1.1) under the given conditions including $|f|^2 \in L_w^{\frac{p}{2}}(\Omega)$ according to the results in [12].

One of the main tools in our approach for proving the main theorem is the Hardy–Littlewood maximal function which controls the local behavior of a function. For a locally integrable function g defined in $\mathbb{R}^n$, we denote the maximal function of g by

$$\mathcal{M}g(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| dy,$$

at each point $x \in \mathbb{R}^n$. We also use

$$\mathcal{M}_\Omega g = \mathcal{M}(\chi_\Omega g)$$

if g is not defined outside Ω.

We shall use the basic properties of the Hardy–Littlewood maximal function as follows:

(1) (strong p-p estimate)

$$\|\mathcal{M}g\|_{L^p(\mathbb{R}^n)} \leq c\|g\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 < p \leq \infty,$$

where a constant c depends only on n and p.

(2) (weak 1-1 estimate)

$$|\{x \in \mathbb{R}^n : \mathcal{M}g(x) \geq t\}| \leq \frac{c}{t} \|g\|_{L^1(\mathbb{R}^n)} \quad \text{for } \forall\, t > 0,$$

where a constant c depends only on n.

The following is the so-called Muckenhoupt's theorem (see [28] for details). Since $L^s_w(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$ for $1 < s < \infty, \mathcal{M}g$ is meaningful when $g \in L^s_w(\mathbb{R}^n)$.

Lemma 2.5. *Suppose $w \in A_s$ where $1 < s < \infty$. Then there exists a constant $c = c(n, s, [w]_s) > 0$ such that*

$$\int_{\mathbb{R}^n} (\mathcal{M}g)^s w dx \le c \int_{\mathbb{R}^n} |g|^s w dx \tag{2.3}$$

whenever $g \in L^s_w(\mathbb{R}^n)$. Conversely, if (2.3) holds for every $g \in L^s_w(\mathbb{R}^n)$, then $w \in A_s$.

We also need the following standard measure theory from [9].

Lemma 2.6. *Suppose g is a non-negative measurable function in a bounded domain $\Omega \subset \mathbb{R}^n$. Let $\eta > 0$ and $M > 1$ be constants and w be a weight in $\mathbb{R}^n$. Then for $0 < s < \infty$,*

$$g \in L^s_w(\Omega) \quad \textit{if and only if } S := \sum_{k \ge 1} M^{sk} w(\{x \in \Omega : g(x) > \eta M^k\}) < \infty$$

and moreover

$$c^{-1} S \le \|g\|^s_{L^s_w(\Omega)} \le c(w(\Omega) + S),$$

where $c > 0$ is a constant depending only on η, M and s.

We next introduce one of the main tools which will be used repeatedly in the proofs of the weighted interior and boundary $W^{2,p}$ estimates.

Lemma 2.7 (Vitali Covering Lemma). *Let $\mathcal{C}$ be a class of balls B_α in $\mathbb{R}^n$ with their radii bounded above. Then there exist disjoint balls $\{B_{\alpha_i}\}_{i=1}^\infty \subset \{B_\alpha\}_\alpha$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_i 5B_{\alpha_i},$$

where $5B_{\alpha_i}$ denotes the ball with the same center as B_{α_i} but with five times the radius.

Indeed, we shall employ the following modified versions of Vitali covering lemma. They can be obtained from the above Vitali covering lemma, see the papers [6, 33] for their proofs and more details.

Lemma 2.8. *Let $0 < \epsilon < 1$, and E and F be measurable sets with $E \subset F \subset B_1$ such that*

(1) $|E| < \epsilon |B_1|$ *and*

(2) *for every* $x \in B_1$ *with* $|E \cap B_r(x)| \geq \epsilon |B_r|, B_r(x) \cap B_1 \subset F$.

Then $|E| \leq 10^n \epsilon |F|$.

Lemma 2.9. *Let* $0 < \epsilon < 1$, *and* E *and* F *be measurable sets with* $E \subset F \subset B_1^+$ *such that*

(1) $|E| < \epsilon |B_1^+|$ *and*
(2) *for every* $x \in B_1^+$ *with* $|E \cap B_r(x)| \geq \epsilon |B_r|, B_r(x) \cap B_1^+ \subset F$.

Then $|E| \leq 2(10^n)\epsilon |F|$.

The following lemma is the weighted version of the modified Vitali covering lemma.

Lemma 2.10. *Let* w *be an* A_s *weight for some* $1 < s < \infty$. *Let* $0 < \epsilon < 1$ *and suppose that the measurable sets* E *and* F *with* $E \subset F \subset B_1^+$ *satisfy the following properties*:

(1) $w(E) < \epsilon w(B_1^+)$, *and*
(2) *for every* $x \in B_1^+$ *and* $0 < r \leq 1$,

$$w(E \cap B_r(x)) \geq \epsilon w(B_r(x)) \quad \textit{implies } B_r(x) \cap B_1^+ \subset F.$$

Then $w(E) \leq 20^{ns} \epsilon [w]_s^2 w(F)$.

Proof. In view of (1), for almost all $x \in E$, there is a small $\rho_x > 0$ such that

$$\begin{aligned} &w(E \cap B_{\rho_x}(x)) = \epsilon w(B_{\rho_x}(x)) \quad \text{and} \\ &w(E \cap B_\rho(x)) < \epsilon w(B_\rho(x)), \quad \forall \rho \in (\rho_x, 1]. \end{aligned} \tag{2.4}$$

Since $\{B_{\rho_x}(x)\}_{x \in E}$ covers E with $\rho_x \leq 1$, the Vitali covering lemma, Lemma 2.7, implies that there is a countable $\{x_i\}_{i=1}^\infty$ so that the balls $B_{\rho_{x_i}}(x_i)$ are mutually disjoint and $E \subset \bigcup_i B_{5\rho_{x_i}}(x_i)$. Then by Lemma 2.3 and (2.4),

$$w(E \cap B_{5\rho_{x_i}}(x_i)) < \epsilon w(B_{5\rho_{x_i}}(x_i)) \leq \epsilon [w]_s 5^{ns} w(B_{\rho_{x_i}}(x_i)).$$

We notice that

$$\sup_{0 < \rho \leq 1} \sup_{x \in B_1^+} \frac{|B_\rho(x)|}{|B_\rho(x) \cap B_1^+|} \leq 4^n.$$

Therefore from Lemma 2.3, we finally obtain

$$\begin{aligned} w(E) &\leq w\left(E \cap \bigcup_{i \geq 1} B_{5\rho_{x_i}}(x_i)\right) \leq \sum_{i \geq 1} \epsilon [w]_s 5^{ns} w(B_{\rho_{x_i}}(x_i)) \\ &\leq [w]_s^2 5^{ns} \epsilon \sum_{i \geq 1} \left(\frac{|B_{\rho_{x_i}}(x_i)|}{|B_{\rho_{x_i}}(x_i) \cap B_1^+|}\right)^s w(B_{\rho_{x_i}}(x_i) \cap B_1^+) \end{aligned}$$

$$\leq [w]_s^2 20^{ns}\epsilon \sum_{i\geq 1} w(B_{\rho_{x_i}}(x_i)\cap B_1^+)$$

$$\leq [w]_s^2 20^{ns}\epsilon w\left(\bigcup_{i\geq 1} B_{\rho_{x_i}}(x_i)\cap B_1^+\right) \leq 20^{ns}\epsilon [w]_s^2 w(F),$$

where the last inequality comes from (2.4) and the second hypothesis. □

We end this section with the following standard iteration lemma (see [19]).

Lemma 2.11. *Let $g : [a,b] \to \mathbb{R}$ be a bounded non-negative function. Assume that for any t, s such that $0 < a \leq t < s \leq b$,*

$$g(t) \leq \eta g(s) + \frac{A}{(s-t)^\alpha} + B,$$

where $A, B \geq 0, \alpha > 0$ and $0 \leq \eta < 1$. Then we have

$$g(t) \leq c\left(\frac{A}{(s-t)^\alpha} + B\right)$$

for some constant $c = c(\alpha, \eta) > 0$.

3. Interior Weighted Estimates

In this section, we shall prove the interior weighted $W^{2,p}$ estimates for the nondivergence type elliptic equation (1.1) via the so-called maximal function approach, which is different from those previously used, for instance, in [11, 22]. We begin with the interior unweighted $W^{2,2}$ estimates for Eq. (1.1) from [11].

Lemma 3.1. *There exists a small $\delta = \delta(\Lambda, n) > 0$ such that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and if $f \in L^2(B_6)$, then for any solution $u \in W^{2,2}(B_6)$ of $a_{ij}D_{ij}u = f$ in B_6, we have the estimate*

$$\|D^2u\|_{L^2(B_1)} \leq c(\|f\|_{L^2(B_6)} + \|u\|_{L^2(B_6)}),$$

where a constant $c > 0$ is independent of u and f.

The following is the main theorem in this section.

Theorem 3.2. *Given $2 < p < \infty$ and a weight $w \in A_{\frac{p}{2}}$, there exists a small $\delta = \delta(\Lambda, p, n, w) > 0$ such that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and if $|f|^2 \in L_w^{\frac{p}{2}}(B_6)$, then for any solution $u \in W^{2,2}(B_6)$ of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_6, \tag{3.1}$$

there holds $|D^2u|^2 \in L_w^{\frac{p}{2}}(B_1)$ and we have the estimate

$$\|D^2u\|_{L_w^p(B_1)} \leq c(\|f\|_{L_w^p(B_6)} + \|u\|_{L^2(B_6)}), \tag{3.2}$$

where a constant $c > 0$ is independent of u and f.

We need the following approximation lemma.

Lemma 3.3. *There is a positive constant $N_1 = N_1(\Lambda, n)$ so that for any $\epsilon > 0$ there exists a small $\delta = \delta(\epsilon, \Lambda, n) > 0$ such that if $u \in W^{2,2}(\Omega)$ is a solution of $a_{ij}D_{ij}u = f$ in $\Omega \supset B_6$ with*

$$\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) \leq 1\} \cap \{x \in \Omega : \mathcal{M}(|f|^2)(x) \leq \delta^2\} \cap B_1 \neq \emptyset \tag{3.3}$$

and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing, then

$$|\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_1| < \epsilon |B_1|.$$

Proof. From the condition (3.3), there is a point $x_0 \in B_1$ such that

$$\frac{1}{|B_\rho|}\int_{B_\rho(x_0)\cap\Omega} |D^2u|^2 dx \leq 1 \quad \text{and} \quad \frac{1}{|B_\rho|}\int_{B_\rho(x_0)\cap\Omega} |f|^2 dx \leq \delta^2,$$

for any $\rho > 0$. Note $B_4 \subset B_5(x_0)$ to see that

$$⨍_{B_4} |D^2u|^2 dx \leq \left(\frac{5}{4}\right)^n ⨍_{B_5(x_0)\cap\Omega} |D^2u|^2 dx \leq 2^n.$$

Likewise, we have that

$$⨍_{B_4} |f|^2 dx \leq 2^n \delta^2.$$

We then use the Poincaré inequality to discover that

$$⨍_{B_4} |Du - (\overline{Du})_{B_4}|^2 dx \leq c ⨍_{B_4} |D^2u|^2 dx \leq c$$

for some positive constant $c = c(n)$.

We next let $v \in W^{2,2}(B_4)$ be the solution of

$$\begin{cases} \overline{a_{ij}}_{B_4} D_{ij}v = 0 & \text{in } B_4, \\ v = u - (\overline{u})_{B_4} - (\overline{Du})_{B_4} \cdot x & \text{on } \partial B_4, \end{cases}$$

to find that for some constant $c = c(n)$,

$$⨍_{B_4} |Dv|^2 dx \leq c ⨍_{B_4} |Du - (\overline{Du})_{B_4}|^2 dx \leq c.$$

We then use the local $C^{1,1}$ estimates to discover

$$\|D^2v\|^2_{L^\infty(B_3)} \leq c ⨍_{B_4} |Dv|^2 dx \leq N_0^2,$$

for some constant $N_0 = N_0(n, \Lambda) > 0$.

Setting $h = u - (\overline{u})_{B_4} - (\overline{Du})_{B_4} \cdot x - v$, we see that $h \in W^{2,2}(B_3)$ is a solution of

$$a_{ij}D_{ij}h = f - (a_{ij} - \overline{a_{ij}}_{B_4})D_{ij}v \quad \text{in } B_3.$$

Then applying Lemma 3.1, we proceed as in Corollary 4.4 to discover that

$$\fint_{B_2} |D^2(u-v)|^2 dx \le c\delta^2,$$

where c is a positive constant depending only on n, Λ.

We next write $N_1 = \max\{4N_0^2, 2^n\}$ and claim

$$\{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \subset \{x \in B_1 : \mathcal{M}_{B_4}(|D^2(u-v)|^2)(x) > N_0^2\}. \tag{3.4}$$

Indeed, suppose $x_1 \in \{x \in B_1 : \mathcal{M}_{B_4}(|D^2(u-v)|^2)(x) \le N_0^2\}$. Then for $\rho \le 2$, $B_\rho(x_1) \subset B_3$ and so

$$\begin{aligned}\fint_{B_\rho(x_1)} |D^2u|^2 dx &\le 2\fint_{B_\rho(x_1)} (|D^2(u-v)|^2 + |D^2v|^2) dx \\ &\le 2\mathcal{M}_{B_4}(|D^2(u-v)|^2)(x_1) + 2N_0^2 \\ &\le 4N_0^2.\end{aligned}$$

On the other hand, if $\rho > 2, x_0 \in B_\rho(x_1) \subset B_{2\rho}(x_0)$, and so we find that

$$\fint_{B_\rho(x_1)} |D^2u|^2 dx \le 2^n \fint_{B_{2\rho}(x_0)} |D^2u|^2 dx \le 2^n.$$

Therefore, $x_1 \in \{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) \le N_1^2\}$, and so the claim (3.4) is proved.

From (3.4) and the weak 1-1 estimate, we finally get

$$\begin{aligned}&\frac{1}{|B_1|}|\{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) > N_1^2\}| \\ &\quad\le \frac{1}{|B_1|}|\{x \in B_1 : \mathcal{M}_{B_4}(|D^2(u-v)|^2)(x) > N_0^2\}| \\ &\quad\le c\fint_{B_2} |D^2(u-v)|^2 dx \le c\delta^2 < \epsilon\end{aligned}$$

by taking δ satisfying the last inequality above, with c being depending only on n, Λ. □

With the approximation lemma above, we have its weighted version whose proof is similar to that of Lemma 4.6.

Lemma 3.4. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty, y \in \Omega$ and $r > 0$. Then there is a constant $N_1(n, \Lambda) > 0$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon, \Lambda, n, w, s) > 0$ such that if $u \in W^{2,2}(\Omega)$ is a solution of $a_{ij}D_{ij}u = f$ in $\Omega \supset B_{6r}(y)$ with*

$$\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) \le 1\} \cap \{x \in \Omega : \mathcal{M}(|f|^2)(x) \le \delta^2\} \cap B_r(y) \ne \emptyset$$

and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6r)$-vanishing, then we have

$$w(\{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r(y)) < \epsilon w(B_r(y)).$$

By a scaling argument, we now have the following lemma.

Lemma 3.5. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$. Then there is a constant $N_1 = N_1(n,\Lambda) > 0$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon,\Lambda,n,w,s) > 0$ such that if $u \in W^{2,2}(\Omega)$ is a solution of $a_{ij}D_{ij}u = f$ in $\Omega \supset B_6$ with*

$$w(\{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r(y)) \geq \epsilon w(B_r(y))$$

for all $y \in B_1$ and for all $r \in (0, \frac{1}{2})$, and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing, then we have

$$B_r(y) \cap B_1 \subset \{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) > 1\} \cup \{x \in B_1 : \mathcal{M}(|f|^2)(x) > \delta^2\}.$$

In view of Lemma 2.8, we derive the following power decay estimate. We refer to the proof of Lemma 4.8 for its completeness.

Lemma 3.6. *Under the same assumptions as in Lemma* 3.5, *we further assume*

$$w(\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_1) < \epsilon w(B_1).$$

Then we have

$$\begin{aligned} &w(\{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) > N_1^{2k}\}) \\ &\quad \leq \epsilon_1^k w(\{x \in B_1 : \mathcal{M}(|D^2u|^2)(x) > 1\}) \\ &\qquad + \sum_{i=1}^{k} \epsilon_1^i w(\{x \in B_1 : \mathcal{M}(|f|^2)(x) > \delta^2 N_1^{2(k-i)}\}), \end{aligned}$$

where $\epsilon_1 = 10^{ns}\epsilon[w]_s^2$.

We are now ready to prove the main theorem of this section. Let us take N_1, ϵ and the corresponding δ to be the same as in the previous lemma.

Proof of Theorem 3.2. In this proof, we denote c to mean a universal constant which can be computed in terms of n, Λ, p and w. From the assumptions that $|f|^2 \in L_w^{\frac{p}{2}}(B_6), w \in A_{\frac{p}{2}}$ and Hölder inequality, we find

$$\int_{B_6} |f|^2 dx \leq \left(\int_{B_6} |f|^p w dx\right)^{\frac{2}{p}} (w^{\frac{-2}{p-2}}(B_6))^{\frac{p-2}{p}} \leq c\|f\|^2_{L_w^p(B_6)}, \tag{3.5}$$

and so $|f| \in L^2(B_6)$. Then by Lemma 3.1, there exists a unique solution u of (3.1) with the estimate

$$\|D^2u\|_{L^2(B_1)} \leq c\left(\|f\|_{L^2(B_6)} + \|u\|_{L^2(B_6)}\right). \tag{3.6}$$

We consider $\tilde{u} = \frac{\delta u}{(\|f\|_{L_w^p(B_6)} + \|u\|_{L^2(B_6)})}$ and $\tilde{f} = \frac{\delta f}{(\|f\|_{L_w^p(B_6)} + \|u\|_{L^2(B_6)})}$. Observe that $\tilde{u} \in W^{2,2}(B_6)$ is a solution of

$$a_{ij}D_{ij}\tilde{u} = \tilde{f} \quad \text{in } B_6$$

with $\|\tilde{f}\|_{L^2(B_6)} + \|\tilde{u}\|_{L^2(B_6)} \leq c\|\tilde{f}\|_{L^p_w(B_6)} + \|\tilde{u}\|_{L^2(B_6)} \leq c\delta$. Then it follows from (3.5), (3.6) and the weak 1-1 estimate that

$$\frac{1}{|B_1|}|\{x \in B_1 : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^2\}| \\ \leq c \fint_{B_1} |D^2\tilde{u}|^2 dx \leq c\left(\fint_{B_6} |\tilde{f}|^2 dx + \fint_{B_6} |\tilde{u}|^2 dx\right) \leq c\delta^2.$$

We then recall Lemma 2.3 to discover that

$$\frac{1}{w(B_1)} w(\{x \in B_1 : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^2\}) \\ \leq \beta\left(\frac{|\{x \in B_1 : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^2\}|}{|B_1|}\right)^\nu \leq c\beta\delta^{2\nu} < \epsilon,$$

by taking δ in order to get the last inequality. Thus we are under the hypotheses of Lemma 3.6. We recall Lemmas 2.5 and 2.6 to observe that

$$\sum_{k=1}^\infty N_1^{pk} w(\{x \in B_1 : \mathcal{M}(|\tilde{f}|^2)(x) > \delta^2 N_1^{2k}\}) \leq c\left\|\frac{\tilde{f}}{\delta}\right\|^p_{L^p_w(B_6)} \leq c.$$

Then by Lemma 3.6, we have

$$\begin{aligned}
&\sum_{k=1}^\infty N_1^{pk} w(\{x \in B_1 : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^{2k}\}) \\
&\quad \leq \sum_{k=1}^\infty N_1^{pk}\left\{\epsilon_1^k w(\{x \in B_1 : \mathcal{M}(|D^2\tilde{u}|^2)(x) > 1\})\right. \\
&\qquad \left. + \sum_{i=1}^k \epsilon_1^i w(\{x \in B_1 : \mathcal{M}(|\tilde{f}|^2)(x) > \delta^2 N_1^{2(k-i)}\})\right\} \\
&\quad = \sum_{k=1}^\infty N_1^{pk}\epsilon_1^k w(\{x \in B_1 : \mathcal{M}(|D^2\tilde{u}|^2)(x) > 1\}) \\
&\qquad + \sum_{i=1}^\infty (N_1^p\epsilon_1)^i \left(\sum_{k=i}^\infty N_1^{p(k-i)} w(\{x \in B_1 : \mathcal{M}(|\tilde{f}|^2)(x) > \delta^2 N_1^{2(k-i)}\})\right) \\
&\quad \leq \sum_{k=1}^\infty (N_1^p\epsilon_1)^k (w(B_1) + c).
\end{aligned}$$

We now take ϵ_1 so that $N_1^p\epsilon_1 < 1$, and then we conclude from Lemmas 2.5 and 2.6 that $\|D^2\tilde{u}\|_{L^p_w(B_1)} \leq c^*$ for some positive constant $c^* = c^*(\Lambda, n, p, w)$. We return from $\tilde{u}$ to u and make a standard procedure for higher integrability for u, to finally derive the desired estimate (3.2). □

4. Weighted Estimates on the Flat Domain

In this section, we derive a weighted $W^{2,p}$ estimate on the flat boundary. To this end, we consider a special case that the domain under consideration is a half ball. We start with an unweighted $W^{2,2}$ estimate near the flat boundary from [12].

Lemma 4.1. *There exists a small $\delta = \delta(\Lambda, n) > 0$ so that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and if $f \in L^2(B_6^+)$, then for any solution $u \in W^{2,2}(B_6^+)$ of*

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } B_6^+, \\ u = 0 & \text{on } T_6, \end{cases}$$

we have the estimate

$$\|D^2u\|_{L^2(B_1^+)} \leq c(\|f\|_{L^2(B_6^+)} + \|u\|_{L^2(B_6^+)}),$$

where a constant $c > 0$ is independent of u and f.

We now state the main theorem in this section.

Theorem 4.2. *Given $2 < p < \infty$ and a weight $w \in A_{\frac{p}{2}}$, there exists a small $\delta = \delta(\Lambda, p, n, w) > 0$ so that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and $|f|^2 \in L_w^{\frac{p}{2}}(B_6^+)$, then a solution $u \in W^{2,2}(B_6^+)$ of*

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } B_6^+, \\ u = 0 & \text{on } T_6, \end{cases} \tag{4.1}$$

satisfies $|D^2u|^2 \in L_w^{\frac{p}{2}}(B_1^+)$ with the estimate

$$\|D^2u\|_{L_w^p(B_1^+)} \leq c(\|f\|_{L_w^p(B_6^+)} + \|u\|_{L^2(B_6^+)}), \tag{4.2}$$

where a constant $c > 0$ is independent of u and f.

The following is a key lemma to prove the above main result in this section.

Lemma 4.3. *For any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon, n, \Lambda) > 0$ so that if $u \in W^{2,2}(B_6^+)$ is a solution of*

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } B_6^+, \\ u = 0 & \text{on } T_6, \end{cases} \tag{4.3}$$

with

$$\fint_{B_4^+} |D^2u|^2 dx \leq 1 \quad \text{and} \quad \fint_{B_4^+} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4^+}|^2 dx \leq \delta^2,$$

then there exists a solution $v \in W^{2,2}(B_4^+)$ of

$$\begin{cases} \overline{a_{ij}}_{B_4^+} D_{ij}v = 0 & \text{in } B_4^+, \\ v = 0 & \text{on } T_4, \end{cases} \tag{4.4}$$

with

$$\fint_{B_4^+} |D^2 v|^2 dx \le 1$$

such that

$$\int_{B_4^+} |u - (\overline{D_n u})_{B_4^+} x_n - v|^2 dx \le \epsilon^2.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $\{u_k\}_{k=1}^{\infty}$, $\{f_k\}_{k=1}^{\infty}$ and $\{\mathbf{A}_k\}_{k=1}^{\infty} = \{(a_{ij}^k)\}_{k=1}^{\infty}$ such that $u_k \in W^{2,2}(B_6^+)$ is a solution of

$$\begin{cases} a_{ij}^k D_{ij} u_k = f_k & \text{in } B_6^+, \\ u_k = 0 & \text{on } T_6, \end{cases}$$

with

$$\fint_{B_4^+} |D^2 u_k|^2 dx \le 1 \quad \text{and} \quad \fint_{B_4^+} |f_k|^2 + |\mathbf{A}_k - \overline{\mathbf{A}_k}_{B_4^+}|^2 dx \le \frac{1}{k^2}, \tag{4.5}$$

but

$$\int_{B_4^+} |u_k - (\overline{D_n u_k})_{B_4^+} x_n - v|^2 dx > \epsilon_0^2, \tag{4.6}$$

for any solution $v \in W^{2,2}(B_4^+)$ of (4.4) satisfying

$$\fint_{B_4^+} |D^2 v|^2 dx \le 1.$$

We write $w_k := u_k - (\overline{D_n u_k})_{B_4^+} x_n$ and claim

$$\|w_k\|_{W^{2,2}(B_4^+)} \le c \tag{4.7}$$

for some positive constant $c = c(n, \Lambda)$. To do this, recalling $D_i u_k = 0$ on T_4 for $1 \le i \le n-1$, we use Poincaré inequality and (4.5) to find that for some $c = c(n) > 0$,

$$\fint_{B_4^+} |D_i(w_k)|^2 dx \le c \fint_{B_4^+} |D_i(u_k)|^2 dx \le c \fint_{B_4^+} |D^2 u_k|^2 dx \le c$$

for $1 \le i \le n-1$. Moreover, we see that

$$\fint_{B_4^+} |D_n(w_k)|^2 dx = \fint_{B_4^+} |D_n u_k - (\overline{D_n u_k})_{B_4^+}|^2 dx \le c \fint_{B_4^+} |D^2 u_k|^2 dx \le c$$

for some constant $c = c(n) > 0$. Thus, we have that for some positive constant $c = c(n, \Lambda)$,

$$\fint_{B_4^+} |D w_k|^2 dx \le c. \tag{4.8}$$

But then, since $w_k = 0$ in T_4, it follows from the Poincaré inequality and (4.8) that for some $c = c(n) > 0$,

$$\fint_{B_4^+} |w_k|^2 dx \le c \fint_{B_4^+} |Dw_k|^2 dx \le c. \tag{4.9}$$

We next recall (4.5) to see that

$$\fint_{B_4^+} |D^2 w_k|^2 dx = \fint_{B_4^+} |D^2 u_k|^2 dx \le 1. \tag{4.10}$$

Then the claim (4.7) follows from (4.8)–(4.10). Consequently, there exist a subsequence of $\{w_k\}_{k=1}^{\infty}$, which we still denote by $\{w_k\}_{k=1}^{\infty}$, and a function $w_0 \in W^{2,2}(B_4^+)$ such that

$$w_k \rightharpoonup w_0 \ \text{ weakly in } W^{2,2}(B_4^+) \quad \text{and} \quad w_k \to w_0 \text{ strongly in } L^2(B_4^+). \tag{4.11}$$

In addition, it follows from (4.10) and (4.11) that

$$\fint_{B_4^+} |D^2 w_0|^2 dx \le 1. \tag{4.12}$$

Since $\{\overline{\mathbf{A}_k}_{B_4^+}\}$ is uniformly bounded in $L^\infty(B_4^+)$, it also has a subsequence, which is denoted by $\{\overline{\mathbf{A}_k}\}$, such that $\|\overline{\mathbf{A}_k} - \mathbf{A}_0\|_{L^\infty(B_4^+)} \to 0$ as $k \to \infty$ for some constant matrix $\mathbf{A}_0 = (a_{ij}^0)$. Then by (4.5), we have

$$\mathbf{A}_k \to \mathbf{A}_0 \quad \text{in } L^2(B_4^+). \tag{4.13}$$

From (4.5), (4.11) and (4.13), it is easy to check that $w_0 \in W^{2,2}(B_4^+)$ is a solution of

$$\begin{cases} a_{ij}^0 D_{ij} w_0 = 0 & \text{in } B_4^+, \\ w_0 = 0 & \text{on } T_4. \end{cases}$$

We then recall (4.11) and (4.12) to reach a contradiction to the inequality (4.6). This completes the proof. □

Corollary 4.4. *Under the hypotheses and conclusion of Lemma* 4.3, *we have*

$$\fint_{B_2^+} |D^2(u - v)|^2 dx \le \epsilon^2.$$

Proof. We apply Lemma 4.3 to η and $\delta(\eta, n, \Lambda)$ replaced by ϵ and $\delta(\epsilon, n, \Lambda)$ respectively, to find that there is a solution $v \in W^{2,2}(B_4^+)$ of (4.4) such that

$$\fint_{B_4^+} |D^2 v|^2 dx \le 1 \quad \text{and} \quad \int_{B_4^+} |u - (\overline{D_n u})_{B_4^+} x_n - v|^2 dx \le \eta^2,$$

provided that

$$\fint_{B_4^+} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4^+}|^2 dx \le \delta^2.$$

Then by $C^{1,1}$ regularity for (4.4) up to the flat boundary, we discover that

$$\|D^2 v\|^2_{L^\infty(B_3^+)} \le c \fint_{B_4^+} |D^2 v|^2 dx \le c,$$

for some positive constant $c = c(n, \Lambda)$.

We next observe that $h = u - (\overline{D_n u})_{B_4^+} x_n - v \in W^{2,2}(B_3^+)$ is a solution of

$$\begin{cases} a_{ij} D_{ij} h = f - (a_{ij} - \overline{a_{ij}}_{B_4^+}) D_{ij} v & \text{in } B_3^+, \\ h = 0 & \text{on } T_3. \end{cases}$$

Then according to Lemma 4.1, we compute for some constant $c = c(n, \Lambda) > 0$,

$$\begin{aligned} \fint_{B_2^+} |D^2(u - v)|^2 dx &\le c \left(\fint_{B_3^+} |f - (a_{ij} - \overline{a_{ij}}_{B_4^+}) D_{ij} v|^2 dx \right. \\ &\quad \left. + \fint_{B_3^+} |u - (\overline{D_n u})_{B_4^+} x_n - v|^2 dx \right) \\ &\le c \left(\fint_{B_4^+} |f|^2 dx + \|D^2 v\|^2_{L^\infty(B_3^+)} \fint_{B_4^+} |a_{ij} - \overline{a_{ij}}_{B_4^+}|^2 \, dx \right. \\ &\quad \left. + \int_{B_4^+} |u - (\overline{D_n u})_{B_4^+} x_n - v|^2 dx \right) \\ &\le c(\delta^2 + \eta^2) \le \epsilon^2, \end{aligned}$$

if we take η and δ satisfying the last inequality. This finishes the proof. □

Lemma 4.5. *There is a positive constant $N_1 = N_1(n, \Lambda)$ so that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, \Lambda, n) > 0$ such that if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of*

$$\begin{cases} a_{ij} D_{ij} u = f & \text{in } \Omega \supset B_6^+, \\ u = 0 & \text{on } \partial\Omega \supset T_6, \end{cases} \tag{4.14}$$

with

$$B_1^+ \cap \{x \in \Omega : \mathcal{M}(|D^2 u|^2)(x) \le 1\} \cap \{x \in \Omega : \mathcal{M}(|f|^2)(x) \le \delta^2\} \ne \emptyset, \tag{4.15}$$

and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing, then there holds

$$\left|\{x \in \Omega : \mathcal{M}(|D^2 u|^2)(x) > N_1^2\} \cap B_1^+\right| < \epsilon |B_1^+|.$$

Proof. From the hypothesis (4.15), there exists a point $x_0 \in B_1^+$ so that

$$\frac{1}{|B_\rho|} \int_{B_\rho^+(x_0) \cap \Omega} |D^2 u|^2 dx \le 1 \quad \text{and} \quad \frac{1}{|B_\rho|} \int_{B_\rho^+(x_0) \cap \Omega} |f|^2 dx \le \delta^2 \quad \text{for all } \rho > 0.$$

Since $B_4^+ \subset B_5^+(x_0)$, we can get

$$\fint_{B_4^+} |D^2u|^2 dx \leq \left(\frac{5}{4}\right)^n \fint_{B_5^+(x_0)} |D^2u|^2 dx \leq 2^n$$

and similarly

$$\fint_{B_4^+} |f|^2 dx \leq 2^n \delta^2.$$

Let us apply Corollary 4.4 to Eq. (4.14) with u and f replaced by $(\frac{1}{2^{\frac{n}{2}}})u$ and $(\frac{1}{2^{\frac{n}{2}}})f$ respectively, in order to have that for any $\eta > 0$, there exist a small $\delta = \delta(\eta) > 0$, a positive constant $N_0 = N_0(n, \Lambda)$ and a solution $v \in W^{2,2}(B_4^+)$ of

$$\begin{cases} \overline{a_{ij}}_{B_4^+} D_{ij} v = 0 & \text{in } B_4^+, \\ v = 0 & \text{on } T_4 \end{cases}$$

such that

$$\|D^2 v\|^2_{L^\infty(B_3^+)} \leq N_0^2 \quad \text{and} \quad \fint_{B_2^+} |D^2(u - v)|^2 dx \leq \eta^2,$$

provided that

$$\fint_{B_4^+} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4^+}|^2 dx \leq \delta^2.$$

Then we can now show in almost the same way as we did in the proof of Lemma 3.3 that

$$\{x \in B_1^+ : \mathcal{M}(|D^2u|^2) > N_1^2\} \subset \{x \in B_1^+ : \mathcal{M}_{B_4^+}(|D^2(u - v)|^2) > N_0^2\}, \tag{4.16}$$

where $N_1^2 := \max\{4N_0^2, 2^n\}$. So we discover that for some $c = c(n, \Lambda) > 0$,

$$\begin{aligned} &\frac{1}{|B_1^+|} |\{x \in B_1^+ : \mathcal{M}(|D^2u|^2) > N_1^2\}| \\ &\quad \leq \frac{1}{|B_1^+|} |\{x \in B_1^+ : \mathcal{M}_{B_4^+}(|D^2(u - v)|^2) > N_0^2\}| \\ &\quad \leq c \fint_{B_2^+} |D^2(u - v)|^2 dx \leq c\eta^2 < \epsilon, \end{aligned}$$

if we take η and δ satisfying the last inequality above. $\square$

Lemma 4.6. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$. There is a positive constant $N_1 = N_1(\Lambda, n)$ so that for any $\epsilon > 0$ and for every $0 < r \leq 1$, there exists a small $\delta = \delta(\epsilon, \Lambda, n, w, s) > 0$ such that if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of*

$$\begin{cases} a_{ij} D_{ij} u = f & \text{in } \Omega \supset B_{6r}^+, \\ u = 0 & \text{on } \partial\Omega \supset T_{6r}, \end{cases}$$

with

$$B_r^+ \cap \{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) \leq 1\} \cap \{x \in \Omega : \mathcal{M}(|f|^2)(x) \leq \delta^2\} \neq \emptyset, \quad (4.17)$$

and if $\mathbf{A}$ *is uniformly elliptic and* $(\delta, 6r)$*-vanishing, then*

$$w(\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r^+) < \epsilon w(B_r^+).$$

Proof. Let us first define $\tilde{a}_{ij}(x) = a_{ij}(rx), \tilde{u}(x) = \frac{1}{r^2}u(rx), \tilde{f}(x) = f(rx)$ and $\tilde{\Omega} = \{\frac{1}{r}x : x \in \Omega\}$. Then we note that $\tilde{u} \in W^{2,2}(\tilde{\Omega}) \cap W_0^{1,2}(\tilde{\Omega})$ is the solution of

$$\begin{cases} \tilde{a}_{ij}D_{ij}\tilde{u} = \tilde{f} & \text{in } \tilde{\Omega} \supset B_6^+, \\ \tilde{u} = 0 & \text{on } \partial\tilde{\Omega} \supset T_6. \end{cases}$$

Let $\epsilon > 0$ be given and choose $\delta = \delta(\epsilon, \Lambda, n, w, s)$ as in Lemma 4.5 with ϵ replaced by $(\frac{\epsilon}{2\beta})^{\frac{1}{\nu}}$, where β and ν are the constants in Lemma 2.3. From (4.17), there exists $x_0 \in B_r^+ \cap \{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) \leq 1\} \cap \{x \in \Omega : \mathcal{M}(|f|^2)(x) \leq \delta^2\}$. Then $z_0 := \frac{1}{r}x_0 \in B_1^+ \cap \{z \in \tilde{\Omega} : \mathcal{M}(|D^2\tilde{u}|^2)(z) \leq 1\} \cap \{z \in \tilde{\Omega} : \mathcal{M}(|\tilde{f}|^2)(z) \leq \delta^2\}$. Since all the hypotheses of Lemma 4.5 are satisfied, Lemma 4.5 gives

$$|\{z \in \tilde{\Omega} : \mathcal{M}(|D^2\tilde{u}|^2)(z) > N_1^2\} \cap B_1^+| < \left(\frac{\epsilon}{2\beta}\right)^{\frac{1}{\nu}} |B_1^+|.$$

Then

$$|\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r^+| < \left(\frac{\epsilon}{2\beta}\right)^{\frac{1}{\nu}} |B_r^+|. \quad (4.18)$$

Using Lemma 2.3, we finally get from (4.18) that

$$\begin{aligned} &w\left(\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r^+\right) \\ &\quad \leq \beta \left(\frac{|\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r^+|}{|B_r^+|}\right)^{\nu} w(B_r^+) \\ &\quad \leq \frac{\epsilon}{2} w(B_r^+) < \epsilon w(B_r^+). \end{aligned}$$

□

Lemma 4.7. *Let* w *be an* A_s *weight in* $\mathbb{R}^n$ *for some* $1 < s < \infty$. *Then there is a constant* $N_1 = N_1(\Lambda, n, w) > 0$ *so that for any* $\epsilon > 0, 0 < r \leq \frac{1}{18}$ *and* $y \in B_1^+$, *there exists a small* $\delta = \delta(\epsilon, n, \Lambda, w) > 0$ *such that if* $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ *is a solution of*

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } \Omega \supset B_6^+, \\ u = 0 & \text{on } \partial\Omega \supset T_6, \end{cases}$$

with

$$w(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r(y)) \geq \epsilon w(B_r(y)) \quad (4.19)$$

and if $\mathbf{A}$ *is uniformly elliptic and* $(\delta, 6)$*-vanishing, then*

$$B_r(y) \cap B_1^+ \subset \{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > 1\} \cup \{x \in B_1^+ : \mathcal{M}(|f|^2)(x) > \delta^2\}. \tag{4.20}$$

Proof. We prove it by contradiction. To do this, assume that (4.19) holds and the conclusion (4.20) is false. Then there is a point $x_0 = ({x_0}', x_{0n}) \in B_r(y) \cap B_1^+$ such that

$$\frac{1}{|B_\rho|}\int_{B_\rho^+(x_0)\cap\Omega} |D^2u|^2 dx \le 1 \quad \text{and} \quad \frac{1}{|B_\rho|}\int_{B_\rho^+(x_0)\cap\Omega} |f|^2 dx \le \delta^2,$$

for any $\rho > 0$. If $B_{6r}(x_0) \subset B_6^+$, it can be done from Lemma 3.5. Thus we need only to consider the case $B_{6r}(x_0) \not\subset B_6^+$, which implies $B_{6r}(x_0) \cap T_6 \neq \emptyset$. One can easily check that $({x_0}', 0) \in T_1$ and moreover

$$B_r(y) \cap B_1^+ \subset B_{6r}^+(x_0) \subset B_{12r}^+({x_0}', 0) \subset B_{72r}^+({x_0}', 0) \subset B_6^+ \subset \Omega,$$

for $0 < r \le \frac{1}{18}$. Apply Lemma 4.6 to $B_{12r}^+({x_0}', 0)$ with ϵ replaced by $\frac{2^\nu \epsilon}{\beta[w]_s(\frac{3}{5})^{n\nu}20^{ns}}$, to derive that

$$\begin{aligned} &\frac{1}{w(B_r(y))} w(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r(y)) \\ &\quad\le \frac{1}{w(B_r(y))} w(\{x \in B_{12r}^+({x_0}', 0) : \mathcal{M}(|D^2u|^2)(x) > N_1^2\}) \\ &\quad< \frac{2^\nu \epsilon w(B_{12r}^+({x_0}', 0))}{\beta[w]_s(\frac{3}{5})^{n\nu}20^{ns} w(B_r(y))}. \end{aligned}$$

However, Lemma 2.3 implies

$$\begin{aligned} w(B_{12r}^+({x_0}', 0)) &\le \beta \left(\frac{|B_{12r}^+|}{|B_{20r}|}\right)^\nu w(B_{20r}(y)) \\ &\le \beta 2^{-\nu}\left(\frac{3}{5}\right)^{n\nu} [w]_s \left(\frac{|B_{20r}(y)|}{|B_r(y)|}\right)^s w(B_r(y)) \\ &= \beta 2^{-\nu}\left(\frac{3}{5}\right)^{n\nu} [w]_s 20^{ns} w(B_r(y)), \end{aligned}$$

since $B_{12r}^+({x_0}', 0) \subset B_{20r}(y)$. Hence we eventually obtain

$$\frac{1}{w(B_r(y))} w(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r(y)) < \epsilon,$$

which is a contradiction. $\square$

Lemma 4.8. *Let* w *be an* A_s *weight in* $\mathbb{R}^n$ *for some* $1 < s < \infty$ *and let* N_1 *be given by Lemma 4.7. For any* $\epsilon > 0$, *there exists* $\delta = \delta(\epsilon, \Lambda, n, w, s) > 0$ *such that if*

$u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in } \Omega \supset B_6^+, \\ u = 0 & \text{on } \partial\Omega \supset T_6, \end{cases}$$

with

$$w(\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_1^+) < \epsilon w(B_1^+) \tag{4.21}$$

and if $\mathbf{A}$ *is uniformly elliptic and* $(\delta, 6)$*-vanishing, then*

$$\begin{aligned} &w(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^{2k}\}) \\ &\quad \le \epsilon_1^k w(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > 1\}) \\ &\qquad + \sum_{i=1}^{k} \epsilon_1^i w(\{x \in B_1^+ : \mathcal{M}(|f|^2)(x) > \delta^2 N_1^{2(k-i)}\}), \end{aligned}$$

where $\epsilon_1 := 20^{ns}\epsilon[w]_s^2$.

Proof. We use Lemma 2.10 on

$$\begin{aligned} &E := \{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \quad \text{and} \\ &F := \{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > 1\} \cup \{x \in B_1^+ : \mathcal{M}(|f|^2)(x) > \delta^2\}. \end{aligned}$$

From (4.21) and Lemma 4.7, we easily check that E and F satisfy the hypotheses of Lemma 2.10. Then Lemma 2.10 gives $w(E) \le \epsilon_1 w(F)$ with $\epsilon_1 := 20^{ns}\epsilon[w]_s^2$, that is,

$$\begin{aligned} &w(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^2\}) \\ &\quad \le \epsilon_1 w(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > 1\}) \\ &\qquad + \epsilon_1 w(\{x \in B_1^+ : \mathcal{M}(|f|^2)(x) > \delta^2\}). \end{aligned}$$

For any $k \ge 2$, we know

$$E_k := \{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^k\} \subset E,$$

and so $w(E_k) < \epsilon w(B_1^+)$. Therefore for each $\lambda := N_1^{k-1}$, $u_\lambda := \frac{u}{\lambda} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of

$$\begin{cases} a_{ij}D_{ij}u_\lambda = f_\lambda & \text{in } \Omega \supset B_6^+, \\ u_\lambda = 0 & \text{on } \partial\Omega \supset T_6, \end{cases}$$

with $w(E_k^\lambda) < \epsilon w(B_1^+)$, and so

$$\begin{aligned} &w(\{x \in B_1^+ : \mathcal{M}(|D^2u_\lambda|^2)(x) > N_1^2\}) \\ &\quad \le \epsilon_1 w(\{x \in B_1^+ : \mathcal{M}(|D^2u_\lambda|^2)(x) > 1\}) \\ &\qquad + \epsilon_1 w(\{x \in B_1^+ : \mathcal{M}(|f_\lambda|^2)(x) > \delta^2\}), \end{aligned}$$

where $f_\lambda := \frac{f}{\lambda}$ and $E_k^\lambda := \{x \in B_1^+ : \mathcal{M}(|D^2 u_\lambda|^2)(x) > N_1^k\}$. Hence we find

$$\begin{aligned} w(\{x \in B_1^+ &: \mathcal{M}(|D^2 u|^2)(x) > N_1^2\lambda^2\}) \\ &\le \epsilon_1 w(\{x \in B_1^+ : \mathcal{M}(|D^2 u|^2)(x) > \lambda^2\}) \\ &\quad + \epsilon_1 w(\{x \in B_1^+ : \mathcal{M}(|f|^2)(x) > \delta^2\lambda^2\}). \end{aligned}$$

Iterating the foregoing estimate, we finally derive

$$\begin{aligned} w(\{x \in B_1^+ &: \mathcal{M}(|D^2 u|^2)(x) > N_1^{2k}\}) \\ &\le \epsilon_1^k w(\{x \in B_1^+ : \mathcal{M}(|D^2 u|^2)(x) > 1\}) \\ &\quad + \sum_{i=1}^{k} \epsilon_1^i w(\{x \in B_1^+ : \mathcal{M}(|f|^2)(x) > \delta^2 N_1^{2(k-i)}\}), \end{aligned}$$

for any positive integer k. $\square$

Now we are ready to give a proof of Theorem 4.2. Let us take N_1, ϵ and the corresponding δ given by the previous lemma. Hereafter we employ c to denote any constant that can be computed in terms of n, Λ, p and w.

Proof of Theorem 4.2. Since $|f|^2 \in L_w^{\frac{p}{2}}(\Omega)$, we have

$$\int_\Omega |f|^2 dx \le \left(\int_\Omega |f|^p w dx\right)^{\frac{2}{p}} \left(w^{\frac{-2}{p-2}}(\Omega)\right)^{\frac{p-2}{p}} \tag{4.22}$$

by Hölder inequality and so $|f| \in L^2(\Omega)$. Then Lemma 4.1 gives that there is a unique solution $u \in W^{2,2}(B_6^+)$ of (4.1) with the estimate

$$\|D^2 u\|_{L^2(B_1^+)} \le c(\|f\|_{L^2(B_6^+)} + \|u\|_{L^2(B_6^+)}), \tag{4.23}$$

with the constant c independent of u and f.

We write $\tilde{u} = \frac{\delta u}{(\|f\|_{L_w^p(B_6^+)} + \|u\|_{L^2(B_6^+)})}$ and $\tilde{f} = \frac{\delta f}{(\|f\|_{L_w^p(B_6^+)} + \|u\|_{L^2(B_6^+)})}$. Then we see that

$$\|\tilde{f}\|_{L^2(B_6^+)} + \|\tilde{u}\|_{L^2(B_6^+)} \le c\delta,$$

and $\tilde{u} \in W^{2,2}(B_6^+)$ is a solution of

$$\begin{cases} a_{ij} D_{ij}\tilde{u} = \tilde{f} & \text{in } B_6^+, \\ \tilde{u} = 0 & \text{on } T_6. \end{cases}$$

Then by (4.22), (4.23) and the weak 1-1 estimate, we deduce

$$\begin{aligned} &\frac{1}{|B_1^+|}|\{x \in B_1^+ : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^2\}| \\ &\quad \le c\fint_{B_1^+} |D^2\tilde{u}|^2 dx \le c\left(\fint_{B_6^+} |\tilde{f}|^2 dx + \fint_{B_6^+} |\tilde{u}|^2 dx\right) \le c\delta^2 \le \left(\frac{\epsilon}{2\beta}\right)^{\frac{1}{\nu}}, \end{aligned}$$

by taking δ in order to get the last inequality, and hence Lemma 2.3 yields

$$\begin{aligned} & w\left(\{x \in B_1^+ : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^2\}\right) \\ & \quad < \beta\left(\frac{|\{x \in B_1^+ : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^2\}|}{|B_1^+|}\right)^{\nu} w(B_1^+) \\ & \quad \leq \frac{\epsilon}{2} w(B_1^+) < \epsilon w(B_1^+). \end{aligned}$$

Observe $|\tilde{f}|^2 \in L_w^{\frac{p}{2}}(B_6^+)$ with $\|\tilde{f}\|_{L_w^p(B_6^+)} \leq \delta$ and recall Lemmas 2.5 and 2.6, to discover that $\|\mathcal{M}(|\tilde{f}|^2)\|_{L_w^{\frac{p}{2}}(B_6^+)}^{\frac{p}{2}} \leq c\delta^p$, and so

$$\begin{aligned} & \sum_{k=1}^{\infty} N_1^{pk} w(\{x \in B_1^+ : \mathcal{M}(|\tilde{f}|^2)(x) > \delta^2 N_1^{2k}\}) \\ & \quad \leq \sum_{k=1}^{\infty} N_1^{pk} w\left(\left\{x \in B_1^+ : \mathcal{M}\left(\left|\frac{\tilde{f}}{\delta}\right|^2\right)(x) > N_1^{2k}\right\}\right) \\ & \quad \leq c\left\|\mathcal{M}\left(\left|\frac{\tilde{f}}{\delta}\right|^2\right)\right\|_{L_w^{\frac{p}{2}}(B_6^+)}^{\frac{p}{2}} \leq c\left\|\frac{\tilde{f}}{\delta}\right\|_{L_w^p(B_6^+)}^{p} \leq c. \end{aligned}$$

Therefore it follows from Lemma 4.8 that for some $c = c(n, \Lambda, w, p) > 0$,

$$\begin{aligned} & \sum_{k=1}^{\infty} N_1^{pk} w(\{x \in B_1^+ : \mathcal{M}(|D^2\tilde{u}|^2)(x) > N_1^{2k}\}) \\ & \quad \leq \sum_{k=1}^{\infty} N_1^{pk}\Bigg\{\epsilon_1^k w(\{x \in B_1^+ : \mathcal{M}(|D^2\tilde{u}|^2)(x) > 1\}) \\ & \qquad + \sum_{i=1}^{k} \epsilon_1^i w(\{x \in B_1^+ : \mathcal{M}(|\tilde{f}|^2)(x) > \delta^2 N_1^{2(k-i)}\})\Bigg\} \\ & \quad = \sum_{k=1}^{\infty} N_1^{pk}\epsilon_1^k w(\{x \in B_1^+ : \mathcal{M}(|D^2\tilde{u}|^2)(x) > 1\}) \\ & \qquad + \sum_{i=1}^{\infty} (N_1^p\epsilon_1)^i \left(\sum_{k=i}^{\infty} N_1^{p(k-i)} w(\{x \in B_1^+ : \mathcal{M}(|\tilde{f}|^2)(x) > \delta^2 N_1^{2(k-i)}\})\right) \\ & \quad \leq \sum_{k=1}^{\infty} N_1^{pk}\epsilon_1^k (w(B_1^+) + c) \leq c, \end{aligned}$$

by taking ϵ_1 so that $N_1^p\epsilon_1 < 1$. Then we employ once again Lemmas 2.5 and 2.6 to find $\|D^2\tilde{u}\|_{L_w^p(B_1^+)} \leq c$, which in turn implies the desired estimate (4.2). $\square$

5. The Proof of Main Theorem

In this section we shall prove our main result, Theorem 2.4, via standard covering and flattening arguments. To be brief, we first derive the *a priori* weighted $W^{2,p}$ estimate from the interior and boundary estimates which we have obtained in the previous sections. We then remove the *a priori* assumption by an approximation procedure, to complete our proof. Once again we denote by c to mean a universal constant being dependent only on n, Λ, w, p and $\partial\Omega$.

Proof of Theorem 2.4. We start with the *a priori* assumption that

$$u \in W^{2,p}_w(\Omega). \tag{5.1}$$

Fix any point $x_0 \in \partial\Omega$. Since $\partial\Omega \in C^{1,1}$, we assume that

$$\Omega \cap B_r(x_0) = \{x \in \Omega : x_n > \gamma(x')\} \cap B_r(x_0)$$

for some small $r > 0$ and for some $C^{1,1}$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\frac{\partial\gamma}{\partial x_i}(x'_0) = 0$ for any $i = 1, 2, \ldots, n-1$ and $\|\bigtriangledown^2\gamma\|_{L^\infty(\mathbb{R}^{n-1})} < \infty$. We now use change of variables to flatten out the boundary near x_0. To do this, define

$$\begin{cases} y_i = x_i =: \Phi^i(x), & \text{if } i = 1, 2, \ldots, n-1, \\ y_n = x_n - \gamma(x') =: \Phi^n(x), \end{cases}$$

and write $y = \Phi(x)$. We set $\Phi := \Psi^{-1}$ and write $x = \Psi(y)$. Choose $s > 0$ so small that the half ball $B^+_{12s} \subset \Phi(\Omega \cap B_r(x_0))$. Define $\tilde{u}(y) = u(\Psi(y)) = u(x)$ for $y \in B^+_{6s}$ and $\tilde{w}(y) = w(\Psi(y))$ for $y \in \mathbb{R}^n$. Then it can be readily checked that $\tilde{w} \in A_{\frac{p}{2}}$ and $\tilde{u} \in W^{2,2}(B^+_{6s})$ is a solution of

$$\begin{cases} \tilde{a}_{lm} D_{y_l y_m}\tilde{u} = \tilde{f} & \text{in } B^+_{6s}, \\ \tilde{u} = 0 & \text{on } T_{6s}, \end{cases}$$

where

$$\tilde{a}_{lm}(y) = a_{ij}(\Psi(y))\Phi^l_{x_i}(\Psi(y))\Phi^m_{x_j}(\Psi(y)), \quad \text{and}$$

$$\tilde{f}(y) = f(\Psi(y)) - a_{ij}(\Psi(y))\Phi^l_{x_i x_j}(\Psi(y))D_{y_l}\tilde{u}.$$

We now recall the imposed conditions on $\mathbf{A}$ and $\partial\Omega$ and the *a priori* assumption (5.1), to observe that $\tilde{f} \in L^p_{\tilde{w}}(B^+_{6s})$. We also check that the resulting matrix

$$\tilde{\mathbf{A}}(y) = (\tilde{a}_{lm}(y)) = [\bigtriangledown\Phi(\Psi(y))] \cdot \mathbf{A}(\Psi(y)) \cdot [\bigtriangledown\Phi(\Psi(y))]^t$$

satisfies a small BMO assumption. Indeed, note from the conditions on $\mathbf{A}$ and $\partial\Omega$ that

$$\|\tilde{\mathbf{A}}\|_* \le c(\|\mathbf{A}\|_* + \|\bigtriangledown\gamma\|_{L^\infty(B'_r(x'_0))} + \|\bigtriangledown\gamma\|^2_{L^\infty(B'_r(x'_0))}) \le c(\delta + r + r^2),$$

where $B'_\rho(x') := \{y' \in \mathbb{R}^{n-1} : |y' - x'| < \rho\}$ is a ball in $\mathbb{R}^{n-1}$. Then we choose $\delta = \delta(n, \Lambda, \gamma) > 0$ and $r = r(n, \Lambda, \gamma) > 0$ sufficiently small so that all the hypotheses of Theorem 4.2 with $\frac{\tilde{u}(sy)}{s^2}$, $\tilde{\mathbf{A}}(sy)$, $\tilde{f}(sy)$ and $\tilde{w}(sy)$ for $y \in B^+_6$ are satisfied. In turn, we apply Theorem 4.2, and then rescale back, to discover that $|D^2\tilde{u}|^2 \in L^{\frac{p}{2}}_{\tilde{w}}(B_s^+)$ with the estimate

$$\int_{B_s^+} |D^2\tilde{u}|^p \tilde{w}dy \le c\left(\underbrace{\int_{B_{6s}^+} |\tilde{f}|^p \tilde{w}dy}_{I_1} + \underbrace{\frac{1}{s^{2p}}\left[\int_{B_{6s}^+} |\tilde{u}|^2 dy\right]^{\frac{p}{2}}}_{I_2}\right).$$

We recall $\partial\Omega \in C^{1,1}$ and $\tilde{u} \in W^{2,p}_w(\Omega)$, to derive

$$\begin{aligned} I_1 &\le \int_{B_{6s}^+} |f(\Psi)|^p \tilde{w}dy + c\int_{B_{6s}^+} |D\tilde{u}|^p \tilde{w}dy \\ &\le \int_{B_{6s}^+} |f(\Psi)|^p \tilde{w}dy + c\tau s^p \int_{B_{6s}^+} |D^2\tilde{u}|^p \tilde{w}dy + c(\tau,p)\frac{1}{s^p}\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w}dy, \end{aligned}$$

where we have used the weighted Sobolev interpolation inequality for any small $\tau > 0$, see [13]. On the other hand, we recall $\tilde{w} \in A_{\frac{p}{2}}$ and use Hölder's inequality to find

$$\begin{aligned} I_2 &\le \frac{1}{s^{2p}}\left(\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w}dy\right)\left(\int_{B_{6s}^+} \tilde{w}^{\frac{-2}{p-2}} dy\right)^{\frac{p-2}{2}} \\ &\le [\tilde{w}]_{\frac{p}{2}} \frac{|B_{6s}^+|^{\frac{p}{2}}}{\tilde{w}(B_{6s}^+)s^{2p}}\left(\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w}dy\right) \\ &\le [\tilde{w}]^2_{\frac{p}{2}} \frac{|B_6|^{\frac{p}{2}}}{\tilde{w}(B_6)s^{2p}}\left(\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w}dy\right) \\ &\le \frac{c}{s^{2p}}\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w}dy \end{aligned}$$

from a direct computation using Lemma 2.3 that

$$\frac{\tilde{w}(B_{6s}^+)}{\tilde{w}(B_6)} \ge [\tilde{w}]^{-1}_{\frac{p}{2}}\left(\frac{|B_{6s}^+|}{|B_6|}\right)^{\frac{p}{2}}.$$

Consequently, we discover

$$\begin{aligned} \int_{B_s^+} |D^2\tilde{u}|^p \tilde{w}dy &\le c\tau s^p \int_{B_{6s}^+} |D^2\tilde{u}|^p \tilde{w}dy \\ &\quad + c\left(\frac{c(\tau,p)}{s^p} + \frac{1}{s^{2p}}\right)\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w}dy + c\int_{B_{6s}^+} |f(\Psi)|^p \tilde{w}dy \\ &\le c\tau s^p \int_{B_{6s}^+} |D^2\tilde{u}|^p \tilde{w}dy + \frac{c}{s^{2p}}\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w}dy + c\int_{B_{6s}^+} |f(\Psi)|^p \tilde{w}dy. \end{aligned}$$

Take $\tau > 0$ sufficiently small and apply Lemma 2.11 to deduce that for any $\theta \in (0,1)$,

$$\int_{B_{\theta R}^+} |D^2\tilde{u}|^p \tilde{w}dy \le c\left(\int_{B_R^+} |\tilde{f}|^p \tilde{w}dy + \frac{1}{[(1-\theta)R]^{2p}}\int_{B_R^+} |\tilde{u}|^p \tilde{w}dy\right),$$

by letting $R := 6s$. Therefore it follows that

$$\int_{B_s^+} |D^2\tilde{u}|^p \tilde{w} dy \le c\left(\int_{B_{6s}^+} |f(\Psi)|^p \tilde{w} dy + \frac{1}{s^{2p}} \int_{B_{6s}^+} |\tilde{u}|^p \tilde{w} dy\right).$$

Converting back to the x-variables, we conclude

$$\begin{aligned}\int_{V_s} |D^2 u|^p w dx &\le c\left(\int_{\Psi(B_{6s}^+)} |f|^p w dx + \frac{1}{s^{2p}} \int_{\Psi(B_{6s}^+)} |u|^p w dx\right)\\ &\le \frac{c}{s^{2p}}\left(\int_\Omega |f|^p w dx + \int_\Omega |u|^p w dx\right),\end{aligned}$$

where $V_s := \Psi(B_s^+)$. Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ by a finite number of sets $V_{s_1}, V_{s_2}, \ldots, V_{s_N}$ as above and find a finite number of small positive constants $s_1, s_2, \ldots, s_N$. We therefore have, by summing the resulting estimates, along with the interior estimate over some open set $V_{s_0} \Subset \Omega$ so that $\Omega \subset \bigcup_{i=0}^N V_{s_i}$, that

$$|D^2 u|^2 \in L_w^{\frac{p}{2}}(\Omega)$$

with the estimate

$$\int_\Omega |D^2 u|^p w dx \le c\left(\int_\Omega |f|^p w dx + \int_\Omega |u|^p w dx\right).$$

In addition, using the uniqueness of $W^{2,p}$ solutions, we eventually obtain the desired estimate

$$\int_\Omega |D^2 u|^p w dx \le c \int_\Omega |f|^p w dx. \tag{5.2}$$

Now it remains to remove the *a priori* assumption (5.1). To this end, select a sequence $\{a_{ij}^k\}_{k=1}^\infty$ of smooth functions with uniform (δ, R)-vanishing property such that

$$a_{ij}^k \to a_{ij} \quad \text{in } L^t(\Omega) \text{ for each } 1 < t < \infty. \tag{5.3}$$

We also take a sequence $\{f^k\}_{k=1}^\infty$ of smooth functions in $C_0^\infty(\Omega)$ satisfying

$$f^k \to f \quad \text{in } L_w^p(\Omega) \quad \text{and} \quad \|f^k\|_{L_w^p(\Omega)} \le c\|f\|_{L_w^p(\Omega)}. \tag{5.4}$$

Then there exists a unique solution $u^k \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of

$$\begin{cases} a_{ij}^k D_{ij} u^k = f^k & \text{in } \Omega,\\ u^k = 0 & \text{on } \partial\Omega.\end{cases} \tag{5.5}$$

Needless to say, these solutions u^k are in $W_w^{2,p}(\Omega)$. But then from the estimate (5.2), we have

$$\|D^2 u^k\|_{L_w^p(\Omega)} \le c\|f^k\|_{L_w^p(\Omega)}, \tag{5.6}$$

where c is independent of k. Thus it follows from (5.4) and (5.6) that

$$\|D^2 u^k\|_{L_w^p(\Omega)} \le c\|f\|_{L_w^p(\Omega)}. \tag{5.7}$$

On the other hand, we recall the interpolation inequality in [13], and then use the weighted Poincaré inequality, to discover

$$\begin{aligned}\int_\Omega |Du^k|^p w dx &\le \tau \int_\Omega |u^k|^p w dx + c(\tau) \int_\Omega |D^2 u^k|^p w dx \\ &\le c\tau \int_\Omega |Du^k|^p w dx + c(\tau) \int_\Omega |D^2 u^k|^p w dx.\end{aligned}$$

We then select small $\tau > 0$ to derive

$$\|Du^k\|_{L^p_w(\Omega)} \le c\|D^2 u^k\|_{L^p_w(\Omega)} \le c\|f\|_{L^p_w(\Omega)}.$$

This estimate and the weighted Poincaré inequality imply

$$\|u^k\|_{L^p_w(\Omega)} \le c\|Du^k\|_{L^p_w(\Omega)} \le c\|D^2 u^k\|_{L^p_w(\Omega)} \le c\|f\|_{L^p_w(\Omega)},$$

and thus $\{u^k\}_{k=1}^\infty$ is uniformly bounded in $W^{2,p}_w(\Omega)$. Then there exist a subsequence, which we still denote by $\{u^k\}_{k=1}^\infty$, and a function $v \in W^{2,p}_w(\Omega)$ such that

$$u^k \rightharpoonup v \quad \text{weakly in } W^{2,p}_w(\Omega). \tag{5.8}$$

Consequently, it follows from (5.7) that

$$\|D^2 v\|_{L^p_w(\Omega)} \le \liminf_{k\to\infty} \|D^2 u^k\|_{L^p_w(\Omega)} \le c\|f\|_{L^p_w(\Omega)}. \tag{5.9}$$

In view of (5.3)–(5.5) and (5.8), we easily observe that v is also a solution of (1.1). Then by the uniqueness for the problem (1.1) we conclude $u = v$. Hence the proof is completed by (5.9). □

Acknowledgments

The authors are sincerely grateful to Professor Lihe Wang for his helpful discussions and valuable comments throughout all this work. S. Byun was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (2009-0083521).

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