2017 Mathematics Placement Test
(Feb. 14, 2017, Time: 90 min.)

★ For questions 1-11, write answers only. ★

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A. Basic (3 points for each, total 18 points)

A-1. The greatest common divisor of x^3-4x^2+3x and $2x^4-5x^3-3x^2$ is a polynomial $\boxed{}$.

A-2. Given two points $A(-2,-1)$ and $B(3,4)$ in $\mathbb{R}^2$, the point $\boxed{}$ divides the line segment AB in the ratio $3:2$.

A-3. $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\boxed{}$.

A-4. For the function $f(x)=\ln(3x-5)$, we have $f'(x)=\boxed{}$.

A-5. In the coordinate plane, the area enclosed by the curve $y=x^2$, the line $x=2$ and the x-axis is $\boxed{}$.

A-6. The line which is perpendicular to the plane $2x+3y-z=1$ and which passes through the point $(1,0,1)$, can be written in the form

$$\frac{x-1}{2}=\frac{y}{a}=\frac{z-1}{b}.$$

Then we have $a+b=\boxed{}$.

B. Intermediate (7 points for each, total 49 points)

B-7. Suppose that a function $f(x)$ satisfies $f(0)=0$ and

$$\lim_{x\to 0}\frac{f(x)\ln(1-2x^2)}{x^3}=8.$$

Then, we have $f'(0)=\boxed{}$.

B-8. Let f be a function defined by

$$f(x)=\int_2^{x^2}\sqrt{t^3+1}\,dt+x^2$$

for $x>0$. The function f has an inverse g which is also differentiable. Then $g'(2)=\boxed{}$.

B-9. $\displaystyle\sum_{n=1}^{\infty}\int_n^{n+1}\frac{2e^{\frac{2}{x}}}{x^2}\,dx=\boxed{}$.

B-10. $\displaystyle\lim_{m\to\infty}\left(\lim_{n\to\infty}\sum_{k=n+1}^{mn}\frac{n}{k^2}\right)=\boxed{}$.

B-11. In the plane, there are two points A and B whose distance apart is 8. A point P is moving along another line l which is parallel to the line AB, with a speed of v per second (here v is a constant satisfying $0<v<5$). Suppose that the point P moved from C to D in two second, where C, D are the points on l which satisfy

$$\overline{AC}+\overline{BC}=10,\ \overline{AD}+\overline{BD}=10.$$

Here C,D are chosen so that $ABDC$ form a quadrilateral. Then, the area of the quadrilateral $ABDC$ as a function of v is $\boxed{}$.

★ **For questions 12-16, show all your works.** ★

B-12. Let f be a continuous function defined on the closed interval $[-1,1]$ which is differentiable on the open interval $(-1,1)$ such that for any $x \in (-1,1)$, we have

$$|f'(x)| \le 1.$$

If in addition we have $f(-1) = -1$ and $f(1) = 1$, then show that $f(x) = x$.

B-13. In the coordinate plane, consider the curve $y = 2\cos x$. Consider the circle which is tangent to this curve at two points $(t, 2\cos t)$, $(-t, 2\cos t)$, and denote the center of the circle as $(0, f(t))$. Compute $\lim_{t\to 0} f(t)$.

C. Advanced (11 points for each, total 33 points)

C-14. Show the following inequality for any positive real numbers a and b

$$(a+b)^{a+b} \le (2a)^a (2b)^b.$$

C-15. Compute the following limit

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n^2} \frac{1}{k}.$$

C-16. Let f be a function defined on an open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ satisfying the following property. For any $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ satisfying $-\frac{\pi}{2} < x + y < \frac{\pi}{2}$, the function f satisfies

$$\big(1 - f(x)f(y)\big) f(x+y) = f(x) + f(y).$$

Answer the following questions

(a) For any $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $-\frac{\pi}{2} < x+y < \frac{\pi}{2}$, show that $f(x)f(y) \ne 1$.

(b) If $\lim_{x\to 0} \frac{f(x)}{x} = 1$, then for any $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ show that the following holds

$$\int_{-a}^{a} (f(x))^2 \, dx = 2f(a) - 2a$$

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