2019 Mathematics Placement Test

(Feb. 12, 2019, Time: 90 min.)

★ For questions 1-11, write answers only. ★

⟨ Space for scratch ⟩

A. Basic Problems (3 points each, total 18 points)

A-1. For A satisfying $\left(\sqrt[3]{3}\right)^{\ln 8} = 3^A$, $\quad A = \boxed{}$.

A-2. For two positive integers A and B satisfying

$$\cos 75^\circ = \frac{\sqrt{A} - \sqrt{B}}{4}, \quad A + B = \boxed{}.$$

A-3. For a differentiable function $f(x)$ such that

$$\lim_{x \to 3} \frac{f(x) - 2}{x - 3} = 3, \quad f(3) + f'(3) = \boxed{}.$$

A-4. For the function $f(x) = x^3 - 12x$ defined on the interval $-3 \leq x \leq 5$, the sum of the maximum value and the minimum value is $\boxed{}$.

A-5. $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\, dx = \boxed{}$.

A-6. The size of the angle between two vectors $\overrightarrow{a} = (2, 1)$ and $\overrightarrow{b} = (1, 3)$ is $\boxed{}$.

B. Intermediate Problems (7 points each, total 49 points)

B-7. The curve $y^2 = 2x^3 + 3x + 3$ and the line $y = 3x + 1$ intersect at three points. Let x_1, x_2 and x_3 be the x-coordinates of the intersection points. Then $\displaystyle\sum_{i=1}^{3} x_i = \boxed{}$ and $\displaystyle\sum_{i=1}^{3} \frac{1}{x_i} = \boxed{}$.

B-8. Suppose that $g(x)$ is the inverse function of a function $f(x) = x^3 + 3x + 1$. Then $\displaystyle\int_1^5 g(x)\, dx = \boxed{}$.

B-9. $\displaystyle\lim_{n \to \infty} \sum_{k=1}^{n^2} \frac{\sqrt{k}}{n^3} = \boxed{}$.

B-10. $\displaystyle\int_0^4 e^{-\sqrt{x}} dx = \boxed{}$.

B-11. Let

$$A(0, 0, 6\sqrt{2}),\ B(2\sqrt{3}, 0, 0),\ C(-\sqrt{3}, 3, 0),\ D(-\sqrt{3}, -3, 0)$$

be four points in the space. Then the radius of the sphere inscribed in the tetrahedron $ABCD$ is $\boxed{}$.

★ For questions 12-16, show all your works. ★

B-12. Determine whether the following statement is true or not. Justify your answer.

"Let $p(x)$ be a polynomial of even degree. Then the graph of $y = p(x)$ has a horizontal tangent line."

B-13. For a positive integer n, answer the following questions.

(a) Show that $\displaystyle\int_1^n \ln x \, dx = n \ln n - (n-1)$.

(b) Calculate $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{n!}$.

C. Advanced Problems (11 points each, total 33 points)

C-14. Let D be a right circular cone of height h and the radius of the base r, and let l be a line passing through the vertex of D and parallel to the base of D. Find the volume of the solid of revolution obtained by rotating D around l.

C-15. Suppose that a function f defined on the real line satisfies the following condition:

For any real number x and $0 < h < \dfrac{1}{2019}$,

$$-4h|x| \le f(x+h) - f(x-h) \le 4h|x|$$

(a) Show that the function f is continuous.

(b) Suppose we have an additional condition $f(1) = 1$. Show that $f(x) \le x^2$ for $x \ge 1$.

C-16. Find the number of expressions consisting of six pairs of parentheses which are 'correctly matched', that is, no initial segment has more close parentheses ')' than open parentheses '('. For example for two pairs, (()) and ()() are correctly matched, while ())(is not.

⟨ Space for scratch ⟩