2009년 2학기 TA 자격 시험: 선형대수학

2009/07/24, 14:30–17:00

1. Construct a 2009×2009 matrix A such that $A^{2009} = O$, but $A^{2008} \neq O$.

2. Find an upper triangular matrix U and a lower triangular matrix L satisfying $LU = A$, where
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}.$$

3. Evaluate the determinants for the following matrices:

   (a) $A = \begin{pmatrix} 2 & -1 & 0 & 1-\epsilon \\ -1 & 2 & -1 & \epsilon - 1 \\ 0 & -1 & 2 & -\epsilon \\ 1-\epsilon & \epsilon-1 & -\epsilon & 2 \end{pmatrix}$, where ϵ is a root of $x^2 - x - 1 = 0$.

   (b) $A = (a_{ij})_{2009\times 2009}$, where $a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$

4. Let A be a square matrix of rank 1. Find a necessary and sufficient condition for diagonalizability of A.

5. Explain why the following two matrices are *not* similar.
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Let V be the real vector space of polynomials in x, y of degree at most 2 and T be the linear operator on V defined by
$$\mathsf{T}(f(x,y)) = \frac{\partial}{\partial x} f(x,y).$$

   (a) Find the characteristic polynomial and the minimal polynomial of T.

   (b) Find the Jordan canonical basis for V to find the Jordan canonical form of T.

7. Let T be a linear operator on a finite-dimensional nonzero vector space V. Prove that V is a T-cyclic subspace of itself if and only if the characteristic polynomial and the minimal polynomial of T coincide up to sign.

8. Let V be an n-dimensional inner product space and S be a maximal orthonormal subset of V. Then show that S consists of exactly n elements.

9. Let V, W be finite-dimensional complex inner product spaces.

   (a) Show that for every linear functional f on V, there exists a unique vector $\mathbf{w} \in \mathsf{V}$ such that $f(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$. Verify that one can define a *conjugate linear* map $\phi_{\mathsf{V}} : \mathsf{V}^* \to \mathsf{V}$ by $\phi_{\mathsf{V}}(f) = \mathbf{w}$.

   (b) Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation. Then prove that T induces the *adjoint map* $\mathsf{T}^* : \mathsf{W} \to \mathsf{V}$ such that

$$\langle \mathsf{T}\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathsf{T}^*\mathbf{w} \rangle.$$

   (c) Prove that the dual map $\mathsf{T}^t : \mathsf{W}^* \to \mathsf{V}^*$ induced by T can be decomposed by

$$\mathsf{T}^t = \phi_{\mathsf{V}}^{-1} \mathsf{T}^* \phi_{\mathsf{W}}.$$

10. (a) Let $\mathscr{M}_n(\mathbb{C})$ be the ring of complex square matrices. Prove that $\mathscr{M}_n(\mathbb{C})$ has no nonzero proper ideal.

   (b) Determine all the $n \times n$ complex matrices X satisfying $AX = XA$ for every $n \times n$ matrix A.
   (Hint for (a), (b): Consider $\{e^{ij}\}$, the standard basis for $\mathscr{M}_n(\mathbb{C})$.)

   (c) Let $\mathsf{V} = \mathscr{M}_n(\mathbb{C})$ with an inner product $\langle A, B \rangle = \operatorname{tr}(AB^*)$, P be a fixed invertible matrix in V, and T_P be the linear operator on V defined by $\mathsf{T}_P(A) = P^{-1}AP$. Find T_P^*, the adjoint operator of T_P.

   (d) Determine a condition for T_P to be self-adjoint.

11. Let B be a non-degenerate symmetric bilinear form defined on a finite-dimensional vector space V, $\mathscr{B} = \{v_i\}$ be a basis for V.

   (a) Prove that there exists a basis $\mathscr{B}' = \{v_i'\}$ for V such that $B(v_i, v_j') = \delta_{ij}$.

   (b) Denote $M_{\mathscr{B}}$ the matrix representation of given bilinear form B. Describe $M_{\mathscr{B}'}$ in terms of $M_{\mathscr{B}}$.