2010년 2학기 TA 자격 시험: 선형대수학

2010/07/26, 14:30–17:00

1. (5×4) Let V, W be finite-dimensional vector spaces. Prove or disprove the following statements.

   (a) If a linear transformation $\mathsf{T}:\mathsf{V}\to\mathsf{W}$ is surjective, then the transpose map $\mathsf{T}^t:\mathsf{W}^*\to\mathsf{V}^*$ induced by T is injective.

   (b) Let $f(t)$ be a non-constant monic polynomial of degree n. Then there exists a linear operator T on n-dimensional space V with the characteristic polynomial $\phi_{\mathsf{T}}(t) = (-1)^n f(t)$.

   (c) Let T be a linear operator on an inner product space V and T^* be its adjoint. Then $[\mathsf{T}^*]_{\mathfrak{B}} = [\mathsf{T}]^*_{\mathfrak{B}}$ for any ordered basis $\mathfrak{B}$.

   (d) Every orthogonal operator on $\mathbb{R}^2$ of determinant 1 is a composition of two reflections about lines.

2. (a) (5) State the Cyclic Decomposition Theorem for primary modules over a principal ideal domain. You don't have to give a proof.

   (b) (10) Let A, B be $n\times n$ square matrices with entries in a field F. Then prove that A, B are similar if and only if F^n_A, F^n_B are isomorphic as $F[t]$-modules, where F^n_A, F^n_B means n-dimensional vector spaces with linear operators $L_A, L_B : F^n \to F^n$, respectively.

   (c) (5) Explain why the following two matrices are *not* similar.

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix},\quad \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$

   (d) (5) Prove that a linear operator T on a finite-dimensional vector space V is diagonalizable if and only if its minimal polynomial is the product of *distinct* linear factors.

3. (5×2)

   (a) Evaluate the determinant of the following matrix

$$A = (a_{ij})_{2010\times 2010},\ \text{where } a_{ij} = \begin{cases} 2 & \text{if } i=j \\ 1 & \text{if } |i-j|=1 \\ 0 & \text{otherwise.}\end{cases}$$

   (b) A square matrix whose columns consist of nonnegative numbers whose sum is 1 is called a *stochastic matrix*. Show that any stochastic matrix has an eigenvalue 1.

4. (a) (10) A self-adjoint linear operator T on a finite-dimensional inner product space V is called *positive semi-definite* if $\langle \mathsf{T}(\mathbf{v}), \mathbf{v} \rangle \geq 0$ for all nonzero $\mathbf{v} \in \mathsf{V}$. Prove that a self-adjoint linear operator T is positive semi-definite if and only if all of its eigenvalues are non-negative.

   (b) (10) Let V and W be finite-dimensional inner product spaces and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation of rank r. Then prove that there exist orthonormal bases $\{v_1, v_2, \ldots, v_n\}$ for V and $\{w_1, w_2, \ldots, w_m\}$ for W and positive scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ such that

$$\mathsf{T}(v_i) = \begin{cases} \sigma_i w_i & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } i > r. \end{cases}$$

   (c) (5) For given matrix A, find a unitary matrix U and a positive semi-definite matrix P satisfying $A = UP$, where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   Verify that this decomposition is *not* unique.

5. Let V be a finite-dimensional vector space with a non-degenerate symmetric bilinear form B.

   (a) (5) Let $\mathsf{f} \in \mathsf{V}^*$ be a linear functional on V. Then show that there exists a unique vector $\mathsf{w} \in \mathsf{V}$ such that $\mathsf{f}(\mathbf{v}) = B(\mathbf{v}, \mathbf{w})$.

   (b) (10) Let W be a subspace of V^* and $\mathsf{W}^0 = \{\mathbf{v} \in \mathsf{V} : f(\mathbf{v}) = 0 \text{ for } f \in \mathsf{W}\}$. Then prove that $\dim \mathsf{V} = \dim \mathsf{W} + \dim \mathsf{W}^0$.

   (c) (5) Let $\mathfrak{B} = \{v_i\}$ be a basis for V. Then prove that there exists a basis $\mathfrak{B}' = \{v_i'\}$ for V such that $B(v_i, v_j') = \delta_{ij}$.