2010 년 1 학기 TA 자격 시험: 선형대수학

2010/01/27, 15:00–17:30

1. (5×2)

   a) Let $\mathsf{V} = \{A \in \mathcal{M}_n(\mathbb{R}) :\ A = A^t\}$, $\mathsf{W} = \{(a_{ij}) \in \mathcal{M}_n(\mathbb{R}) :\ a_{ij} = 0,\ i > j\}$. Determine $\dim(\mathsf{V} + \mathsf{W})$.

   b) Let V be the space of functions defined on $[0, 2\pi]$ generated by $f_k(t) = e^{ikt}$, $(k = 0, 1, \ldots, n)$ over $\mathbb{C}$. Define linear functionals $\varphi_k : \mathsf{V} \to \mathbb{C}$ by

   $$\varphi_k(g) = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-ikt}dt,\ \ k = 0, 1, \ldots, n.$$

   Prove that $\{\varphi_k\}$ forms the dual basis of $\{f_k\}$ for V^*.

2. a) (10) For the given matrix A, evaluate the determinant, and find an upper triangular matrix U and a lower triangular matrix L satisfying $LU = A$, where

   $$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix}.$$

   b) (5) Let $G = GL_n(\mathbb{Z})$, the *group* of invertible matrices with entries in $\mathbb{Z}$. Then show that $G = \{A \in \mathcal{M}_n(\mathbb{Z}) :\ \det(A) = \pm 1\}$.

3. (5×2)

   a) Let T, U be linear operators on V such that $\mathsf{TU}=\mathsf{UT}$. Then show that $\mathsf{N}(\mathsf{U})$, $\mathsf{R}(\mathsf{U})$ are T-invariant.

   b) For any $A \in \mathcal{M}_n(k)$, prove that the characteristic polynomial and the minimal polynomial of A have the same zeros in $\bar{k}$.

4. (8×2) Let V be the real vector space of polynomials in x, y of degree at most 2 and T be the linear operator on V defined by

   $$\mathsf{T}(f(x, y)) = \frac{\partial}{\partial x} f(x, y).$$

   a) Find the characteristic polynomial and the minimal polynomial of T.

   b) Find the Jordan canonical basis for V to find the Jordan canonical form of T.

5. a) (5) A linear operator T on a finite-dimensional inner product space V is called *positive semi-definite* if T is self-adjoint and $\langle \mathsf{T}(\mathbf{v}), \mathbf{v}\rangle \geq 0$ for all nonzero $\mathbf{v} \in \mathsf{V}$. Prove that a self-adjoint linear operator T is positive semi-definite if and only if all of its eigenvalues are non-negative.

   b) (10) Let V and W be finite-dimensional inner product spaces and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation of rank r. Then prove that there exist orthonormal bases $\{v_1, v_2, \ldots, v_n\}$ for V and $\{w_1, w_2, \ldots, w_m\}$ for W and positive scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ such that

$$\mathsf{T}(v_i) = \begin{cases} \sigma_i w_i & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } i > r. \end{cases}$$

6. a) (5) Let V be a finite-dimensional complex inner product space and let g be a linear functional on V. Then show that there exists a unique vector $\mathbf{w} \in \mathsf{V}$ such that $\mathsf{g}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w}\rangle$.

   b) (8) Let V be a finite-dimensional vector space with a non-degenerate bilinear form B, and let $\mathsf{f} \in \mathsf{V}^*$ be a linear functional on V. Then show that there exists a unique vector $\mathsf{w} \in \mathsf{V}$ such that $\mathsf{f}(\mathbf{v}) = B(\mathbf{v}, \mathbf{w})$.

7. (7×3) Let B be a non-degenerate symmetric bilinear form defined on a finite-dimensional vector space V, $\mathcal{B} = \{v_i\}$ be a basis for V.

   a) Prove that there exists a basis $\mathcal{B}' = \{v_i'\}$ for V such that $B(v_i, v_j') = \delta_{ij}$.

   b) Let B be a non-degenerate symmetric bilinear form defined on $\mathbb{R}^2$ with the matrix representation

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

   with respect to the standard ordered basis $\mathcal{B} = \{e_1, e_2\}$. Find the dual basis $\mathcal{B}'$ of $\mathcal{B}$ for $\mathbb{R}^2$ and the matrix representation of B with respect to $\mathcal{B}'$.

   c) Generalize b). That is, denote $M_{\mathcal{B}}$ the matrix representation of the given bilinear form B, then describe $M_{\mathcal{B}'}$ in terms of $M_{\mathcal{B}}$.