2011년 2학기 TA 자격 시험: 선형대수학

2011/07/26, 15:00–17:30

1. (5×3) Let R be a principal ideal domain. Write down the following statements:

   a) Primary decomposition theorem for torsion R-modules.

   b) Cyclic decomposition theorem for p-primary R-modules, where p is prime.

   c) Universal property of the tensor product of R-modules A and B.

2. (5×3) Prove the following statements.

   a) If two square matrices A, B are similar, then $\mathrm{tr}A = \mathrm{tr}B$.

   b) Let F be a field and $A, B \in \mathcal{M}_n(F)$. Prove that A, B are similar if and only if F^n_A, F^n_B are isomorphic as $F[t]$-modules, where F^n_A, F^n_B means n-dimensional vector spaces associated with linear operators $L_A, L_B : F^n \to F^n$, respectively.

   c) Let T be a linear operator on a finite-dimensional vector space V. Then a subspace W of V is T-invariant if and only if it is an $F[t]$-submodule of V.

3. (7) Let V, W be vector spaces over a field F of dimension n, m, respectively. Prove that the dimension of the tensor product $\mathsf{V} \otimes \mathsf{W}$ over F is mn.

4. (7) Let $A = (a_{ij})$ be a 2011×2011 matrix with $a_{ij} = 1$ for $1 \leq i, j \leq 2011$. Then find the minimal polynomial of A.

5. (7×2) Let $\mathsf{T} = \mathsf{L}_A : \mathbb{R}^5 \to \mathbb{R}^5$, where

$$A = \begin{pmatrix} 14 & 8 & -1 & -6 & 2 \\ -12 & -4 & 2 & 8 & -1 \\ 8 & -2 & 0 & -9 & 0 \\ 8 & 8 & 0 & 0 & 2 \\ -8 & -4 & 0 & 4 & 0 \end{pmatrix}.$$

   It's well-known that the minimal polynomial of A is $m(x) = (x-2)^3$.

   a) Find a rational canonical basis for $\mathbb{R}^5$ and a rational canonical form of A.

   b) Find a Jordan canonical basis for $\mathbb{R}^5$ and a Jordan canonical form of A.

6. Let $\mathsf{V} = \mathbb{R}^3$.

   a) (7) Define an inner product on V by

   $$\langle (v_0, v_1, v_2), (w_0, w_1, w_2) \rangle = v_0 w_0 + \frac{1}{3}(v_0 w_2 + v_1 w_1 + v_2 w_0) + \frac{1}{5} v_2 w_2.$$

   Show that this is actually an inner product on V.

   b) (5) Let $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$ be a basis for V. Construct an orthonormal basis $\mathcal{B}'$ from $\mathcal{B}$ with respect to the inner product in a).

7. (5×4) Let V, W be finite-dimensional complex inner product spaces.

   a) Show that for every linear functional f on V, there exists a unique vector $\mathbf{w} \in \mathsf{V}$ such that $f(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$, which induces a *conjugate linear* map $\phi_{\mathsf{V}} : \mathsf{V}^* \to \mathsf{V}$ by $\phi_{\mathsf{V}}(f) = \mathbf{w}$.

   b) Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation. Then prove that T induces the *adjoint map* $\mathsf{T}^* : \mathsf{W} \to \mathsf{V}$ such that

   $$\langle \mathsf{T}\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathsf{T}^*\mathbf{w} \rangle.$$

   c) Let $\mathsf{V} = \mathcal{M}_n(\mathbb{C})$ with an inner product $\langle A, B \rangle = \mathrm{tr}(B^* A)$, P be a fixed invertible matrix in V, and T_P be the linear operator on V defined by $\mathsf{T}_P(A) = P^{-1}AP$. Find T_P^*, the adjoint operator of T_P.

   d) Prove that the adjoint map T^* in b) can be decomposed as

   $$\mathsf{T}^* = \phi_{\mathsf{V}} \mathsf{T}^t \phi_{\mathsf{W}}^{-1},$$

   where $\mathsf{T}^t : \mathsf{W}^* \to \mathsf{V}^*$ is the dual map induced by T.

8. (5×2) Prove the following statements:

   a) Let V be a finite-dimensional vector space with a non-degenerate *symmetric* bilinear form B, and W be its subspace. Then

   $$\dim \mathsf{W} + \dim \mathsf{W}^\perp = \dim \mathsf{V},$$

   where $\mathsf{W}^\perp = \{\mathbf{v} \in \mathsf{V} : B(\mathbf{v}, \mathbf{w}) = 0 \text{ for any } \mathbf{w} \in \mathsf{W}\}$.

   b) Let V be a finite-dimensional vector space with a non-degenerate *alternating* bilinear form B, that is, $B(\mathbf{v}, \mathbf{v}) = 0$ for every $\mathbf{v} \in \mathsf{V}$. Prove that $\dim \mathsf{V}$ must be even.