2011 년 1 학기 TA 자격 시험: 선형대수학

2011/01/24, 15:00–17:30

1. (5×4) Give a brief proof or a counterexample to prove or disprove the following statements.

   a) Every matrix $A \in \mathcal{M}_n(\mathbb{Z})$ is invertible if and only if $\det A = \pm 1$.

   b) Let T be a invertible normal operator on a finite-dimensional complex inner product space V. Then every eigenvalue of T is pure imaginary if and only if $\mathsf{T} + \mathsf{T}^* = 0$.

   c) Let V be a finite-dimensional vector space with a non-degenerate bilinear form B and W be a subspace of V. Then $\mathsf{W} \cap \mathsf{W}^{\perp} = \{0\}$.

   d) Let V be a finite-dimensional vector space with a non-degenerate bilinear form B, and let $\mathsf{f} \in \mathsf{V}^*$ be a linear functional on V. Then show that there exists a unique vector $\mathsf{w} \in \mathsf{V}$ such that $\mathsf{f}(\mathbf{v}) = B(\mathbf{v}, \mathbf{w})$.

2. (5×2)

   a) Construct a 2011×2011 real matrix A such that $A^{2011} = O$, but $A^{2010} \neq O$.

   b) Prove that if a real *symmetric* matrix A satisfies $A^{2011} = O$, then $A = O$.

3. (5×3) Let V, W be vector spaces over a field F of dimension n, m, respectively.

   a) Prove that the dimension of the tensor product $\mathsf{V} \otimes \mathsf{W}$ over F is mn.

   b) Let $\mathsf{T} = \mathsf{L}_A$, $\mathsf{U} = \mathsf{L}_B$ be linear operators on V, W, respectively, which induce the linear operator $\mathsf{T} \otimes \mathsf{U}$ on $\mathsf{V} \otimes \mathsf{W}$. Then determine the matrix representation of $\mathsf{T} \otimes \mathsf{U}$ with respect to the standard (lexicographically) ordered basis.

   c) Determine $\det(\mathsf{T} \otimes \mathsf{U})$. You don't have to justify your answer.

4. a) (5) State the Cyclic Decomposition Theorem for primary modules over a principal ideal domain. You don't have to give a proof.

   b) (8) Consider the following real matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix},$$

   which has the characteristic polynomial $\phi_A(t) = -(t^2+2)^2(t-2)$. Then find a rational canonical basis to determine the rational canonical form of A.

5. (7×2)

   a) Find the spectral decomposition of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

   b) Find a positive definite matrix P and a unitary matrix U satisfying $PU = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$.

      Show that P and U are uniquely determined.

6. Let V be an inner product space and W be a finite-dimensional subspace of V.

   a) (5) Prove that for every $\mathbf{v} \in \mathsf{V}$, there exist unique vectors $\mathbf{w} \in \mathsf{W}$, $\mathbf{w}' \in \mathsf{W}^{\perp}$ satisfying $\mathbf{v} = \mathbf{w} + \mathbf{w}'$.

   b) (8) Find the minimal solution of the equation:

$$\begin{aligned} x + y + 2z &= 6 \\ 2x - z &= 5 \end{aligned}$$

7. (10) Show that every orthogonal operator in $\mathbb{R}^3$ can be decomposed as a composition of at most three reflections about lines.

8. (5) Let B be a non-degenerate symmetric bilinear form defined on $\mathbb{R}^2$ with the matrix representation $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ with respect to the standard ordered basis $\mathcal{B} = \{e_1, e_2\}$. Find the basis $\mathcal{B}'$ for V such that $B(e_i, v_j') = \delta_{ij}$ for $e_i \in \mathcal{B}, v_j \in \mathcal{B}'$. (Hint: Consider the dual basis $\mathcal{B}^*$ of $\mathcal{B}$ for V^*.)