2012년 2학기 TA 자격 시험: 선형대수학

2012/07/26, 15:00–17:30

1. (3×5) State the following theorems. You don't have to give any proof.

   a) Structure theorem for finitely generated modules over a principal ideal domain

   b) Structure theorem for finitely generated free modules over a principal ideal domain

   c) Primary decomposition theorem for finitely generated torsion modules over a principal ideal domain

   d) Cyclic decomposition theorem for finitely generated primary modules over a principal ideal domain

   e) Universal property of the tensor product of R-modules M, N

2. (5×2) Evaluate the determinants of the following matrices:

   a) $A = \begin{pmatrix} 2 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 2 \end{pmatrix}$

   b) $A = (a_{ij})_{2012 \times 2012}$, where $a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$

3. (15) Let $\mathsf{T} = \mathsf{L}_A : \mathbb{Q}^4 \to \mathbb{Q}^4$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

   One can easily verify that the characteristic polynomial of A is $\chi(x) = (x^2 - 2x + 5)^2$. Determine the minimal polynomial of A and find a rational canonical basis for $\mathbb{Q}^4$ and the rational canonical form of A.

4. (10) Let T,U be diagonalizable linear operators on a finite-dimensional vector space V satisfying the relation TU=UT. Then show that T and U are *simultaneously diagonalizable*, that is, they have the common basis consisting of eigenvectors.

5. (5×3)

   a) For any $A \in \mathcal{M}_n(k)$, prove that the characteristic polynomial and the minimal polynomial of A have the same zeros in $\bar{k}$.

   b) Explain why the following two matrices are *not* similar.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   c) Let V be an n-dimensional inner product space and S be a maximal orthonormal subset of V. Then show that S consists of exactly n elements.

6. (5×2) Let V be a finite-dimensional inner product space and T be a linear operator on V.

   a) Prove that if $||\mathsf{T}\mathbf{v}|| = ||\mathsf{T}^*\mathbf{v}||$ for all $\mathbf{v} \in \mathsf{V}$, where V is *complex*, then T is normal.

   b) Prove that if $||\mathsf{T}\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v} \in \mathsf{V}$, where V is *real*, then T is orthogonal, that is, $\mathsf{T}^*\mathsf{T} = \mathsf{T}\mathsf{T}^* = \mathsf{I}_\mathsf{V}$.

7. (5×3) Let V be a finite-dimensional vector space over a field F with a non-degenerate symmetric bilinear form B.

   a) Let $\mathsf{f} \in \mathsf{V}^*$ be a linear functional on V. Then show that there exists a unique vector $\mathsf{w} \in \mathsf{V}$ such that $\mathsf{f}(\mathbf{v}) = B(\mathbf{v}, \mathbf{w})$.

   b) Let $\mathfrak{B} = \{v_i\}$ be a basis for V. Then prove that there exists a basis $\mathfrak{B}' = \{v_i'\}$ for V such that $B(v_i, v_j') = \delta_{ij}$.

   c) Let W be a subspace of V and $\mathsf{W}^\perp = \{\mathbf{v} \in \mathsf{V} : B(\mathbf{v}, \mathbf{w}) = 0 \text{ for } \mathbf{w} \in \mathsf{W}\}$. Then prove that $\dim \mathsf{V} = \dim \mathsf{W} + \dim \mathsf{W}^\perp$.

8. (5×2) Let M, M', N, N' be modules over a commutative ring R, and $f : M \to M'$ and $g : N \to N'$ are R-module homomorphisms. Then prove or disprove the following statements.

   a) If f and g are surjective, so is $f \otimes g : M \otimes N \to M' \otimes N'$.

   b) If f and g are injective, so is $f \otimes g : M \otimes N \to M' \otimes N'$.