2012 년 1 학기 TA 자격 시험: 선형대수학

2012/01/20, 15:00–17:30

1. (3×5) State the following theorems. You don't have to give any proof.

   a) Structure theorem for finitely generated modules over a principal ideal domain

   b) Structure theorem for finitely generated free modules over a principal ideal domain

   c) Primary decomposition theorem for finitely generated torsion modules over a principal ideal domain

   d) Cyclic decomposition theorem for finitely generated primary modules over a principal ideal domain

   e) Universal property of the tensor product of R-modules M, N

2. (5×2)

   a) Find a square matrix X satisfying $X^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   b) Show that there is no square matrix X satisfying $X^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

3. Let A, B be matrices given by:

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -6 & 1 & -6 & 11 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -2 \\ -4 & 0 & -4 & 7 \end{pmatrix}.$$

   a) (5) Find the characteristic polynomial and the minimal polynomial of A.

   b) (8) Give a rational canonical basis to determine a rational canonical form of A over $\mathbb{R}$.

   c) (7) One can verify that $\text{ann}(e_1) = (x^2 - 1)^2$, $\text{ann}(e_2) = (x + 1)^2$, $\text{ann}(e_3) = \text{ann}(e_4) = (x - 1)^2$, where $\text{ann}(e_i)$ means the B-annihilator ideal of each standard unit vector e_i. Then find a Jordan canonical basis and a Jordan canonical form of B.

4. (5) Show that every orthogonal operator in $\mathbb{R}^2$ can be decomposed as a composition of at most two reflections about lines.

5. a) (5) A self-adjoint linear operator T on a finite-dimensional inner product space V is called *positive semi-definite* if $\langle \mathsf{T}(\mathbf{v}), \mathbf{v}\rangle \geq 0$ for all nonzero $\mathbf{v} \in \mathsf{V}$. Prove that a self-adjoint linear operator T is positive semi-definite if and only if all of its eigenvalues are non-negative.

   b) (8) Let V and W be finite-dimensional inner product spaces and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation of rank r. Then prove that there exist orthonormal bases $\{v_1, v_2, \ldots, v_n\}$ for V and $\{w_1, w_2, \ldots, w_m\}$ for W and positive scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ such that

$$\mathsf{T}(v_i) = \begin{cases} \sigma_i w_i & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } i > r. \end{cases}$$

   c) (7) Decompose the following matrix A as a product of some positive definite matrix and some orthogonal matrix, where

$$A = \begin{pmatrix} 11 & -5 \\ -2 & 10 \end{pmatrix}.$$

6. (5×3) Let V be a finite-dimensional vector space over a field F with a non-degenerate bilinear form B.

   a) Let $\mathsf{f} \in \mathsf{V}^*$ be a linear functional on V. Then show that there exists a unique vector $\mathsf{w} \in \mathsf{V}$ such that $\mathsf{f}(\mathbf{v}) = B(\mathbf{v}, \mathbf{w})$.

   b) Let W be a subspace of V and $\mathsf{W}^{\perp} = \{\mathbf{v} \in \mathsf{V} : B(\mathbf{v}, \mathbf{w}) = 0 \text{ for } \mathbf{w} \in \mathsf{W}\}$. Then prove that $\dim \mathsf{V} = \dim \mathsf{W} + \dim \mathsf{W}^{\perp}$.

   c) Let $\mathfrak{B} = \{v_i\}$ be a basis for V. Then prove that there exists a basis $\mathfrak{B}' = \{v_i'\}$ for V such that $B(v_i, v_j') = \delta_{ij}$.

7. (5×3) Let V, W be vector spaces over a field F of dimension n, m, respectively.

   a) Prove that the dimension of the tensor product $\mathsf{V} \otimes \mathsf{W}$ over F is mn.

   b) Let $\mathsf{T} = \mathsf{L}_A$, $\mathsf{U} = \mathsf{L}_B$ be linear operators on V, W, respectively, which induce the linear operator $\mathsf{T} \otimes \mathsf{U}$ on $\mathsf{V} \otimes \mathsf{W}$. Then determine the matrix representation of $\mathsf{T} \otimes \mathsf{U}$ with respect to the standard (lexicographically) ordered basis.

   c) Determine $\det(\mathsf{T} \otimes \mathsf{U})$. You don't have to justify your answer.