2013년 2학기 TA 자격 시험: 선형대수학

2013/07/22, 15:00–17:30

1. State the following theorems. You don't have to give any proof.

   a) Primary decomposition theorem for finitely generated torsion modules over a principal ideal domain

   b) Cyclic decomposition theorem for finitely generated primary modules over a principal ideal domain

2. Let V be a finite-dimensional vector space and W be its subspace. Define

$$\mathsf{W}' = \{f \in \mathsf{V}^* : f(\mathbf{v}) = 0 \text{ for any } \mathbf{v} \in \mathsf{W}\},$$

   which is actually a subspace of V^*.

   a) Prove that $\dim \mathsf{W} + \dim \mathsf{W}' = \dim \mathsf{V}$.

   b) Let T be a linear operator on V. Then show that $\mathsf{N}(\mathsf{T}^t) = \mathsf{R}(\mathsf{T})'$.

3. Let $\mathsf{T} = \mathsf{L}_A : F^4 \to F^4$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

   a) Find the characteristic polynomial of A.

   b) Determine the rational canonical form of A in each case of $F = \mathbb{F}_2$ and $F = \mathbb{F}_3$, where $\mathbb{F}_q$ denotes the finite field of q elements.

4. Prove briefly or give a counterexample to disprove the following statements:

   a) For any $A \in \mathcal{M}_n(k)$, the characteristic polynomial and the minimal polynomial of A have the same zeros in $\bar{k}$.

   b) Every orthogonal operator in $\mathbb{R}^2$ can be decomposed as a composition of at most two relections about lines.

   c) Let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear operator on a finite-dimensional inner product space V and $\mathfrak{B} = \{\mathbf{v}_i\}$ be an orthonormal basis for V. If $||\mathsf{T}(\mathbf{v}_i)|| = ||\mathbf{v}_i||$ holds for any $\mathbf{v}_i \in \mathfrak{B}$, then T is an isometry.

5. Let $\mathsf{V} = \mathcal{M}_2(\mathbb{C})$ and definte an inner product $\langle\ ,\ \rangle$ on V by $\langle A, B\rangle = \mathrm{tr}(B^*A)$.

   a) Let $\mathbb{A}_2 := \{A \in \mathsf{V} : \mathrm{tr}A = 0\}$ be a subspace of V. Construct an orthonormal bases for $\mathbb{A}_2$.

   b) Construct a *nonzero* linear functional $f \in \mathsf{V}^*$ satisfying $f(A) = 0$ for $A \in \langle e^{11} - e^{22}, e^{12}\rangle$.

6. Let V be an inner product space, and W be its finite-dimensional subspace.

   a) Show that $(\mathsf{W}^\perp)^\perp = \mathsf{W}$.

   b) Let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear operator on V. Then prove that $\mathsf{N}(\mathsf{T})^\perp = \mathsf{R}(\mathsf{T}^*)$.

   c) Prove that the maximum number of linearly independent row vectors and the maximum number of linearly independent column vectors of a matrix are same. (Hint: You may use b) of this problem or Problem 2.)

   d) Find the minimal solution of the equation:

$$\begin{aligned} x + y + 2z &= 6 \\ 2x - z &= 5 \end{aligned}$$

   Explain briefly why your solution is unique.

7. Let V,W be finite-dimensional vector spaces over $\mathbb{Q}(i)$ and H_1, H_2 be non-degenerate Hermitian forms on V, W, respectively.

   a) Show that for every linear functional f on V, there exists a unique vector $\mathbf{v}' \in \mathsf{V}$ such that $\mathsf{f}(\mathbf{v}) = H_1(\mathbf{v}, \mathbf{v}')$. And this induces a *conjugate*-linear bijective map $\phi_{\mathsf{V}} : \mathsf{V}^* \to \mathsf{V}$.

   b) Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation. Then show that T induces the *adjoint* map $\mathsf{T}^* : \mathsf{W} \to \mathsf{V}$ satisfying the relation

$$H_2(\mathsf{T}\mathbf{v}, \mathbf{w}) = H_1(\mathbf{v}, \mathsf{T}^*\mathbf{w}).$$

   Show also that T^* is linear.

   c) Prove that the transpose operator $\mathsf{T}^t : \mathsf{W}^* \to \mathsf{V}^*$ of T can be decomposed as

$$\mathsf{T}^t = \phi_{\mathsf{V}}^{-1}\mathsf{T}^*\phi_{\mathsf{W}}.$$

8. Let R be a commutative ring with 1 and M, N be R-modules.

   a) State the universal property of the tensor product of R-modules M and N.

   b) Prove that $R \otimes M \simeq M$ as R-modules.