2013년 1학기 TA 자격 시험: 선형대수학

2013/01/23, 15:00–17:30

1. (5) Find an upper triangular matrix U and a lower triangular matrix L satisfying

$$LU = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}.$$

2. (5×3)

   a) Show that every nonzero ideal of a principal ideal domain R is isomorphic to R as an R-module. Thus it is also a free R-module.

   b) Give an example of an R-submodule of a free R-module, which is not free.

   c) Give an example of an R-module which is torsion-free, but not free.

3. (5×2)

   a) State the Cyclic Decomposition Theorem for primary modules over a principal ideal domain. You don't have to give a proof.

   b) Let A, B be $n \times n$ matrices with entries in a field F. Then prove that A, B are similar if and only if F^n_A, F^n_B are isomorphic as $F[t]$-modules, where F^n_A, F^n_B means n-dimensional vector spaces with linear operators $\mathsf{L}_A, \mathsf{L}_B : F^n \to F^n$, respectively.

4. Let $\mathsf{T} = \mathsf{L}_A : F^4 \to F^4$ $(\mathrm{char} F \neq 2)$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

   One can easily verify that the characteristic polynomial of A is $\chi(x) = (x^2 - 2x + 5)^2$.

   a) (5) Determine the minimal polynomial of A.

   b) (8) Determine the rational canonical form of A in each case of $F = \mathbb{F}_3$ and $F = \mathbb{F}_{13}$, where $\mathbb{F}_q$ denotes the finite field of q elements.

5. Let $\mathbf{v}$ be a nonzero vector in a finite-dimensional vector space V, and k be the smallest positive integer for which

$$\{\mathbf{v}, \mathsf{T}\mathbf{v}, \dots, \mathsf{T}^k\mathbf{v}\}$$

   is linearly dependent, that is, there are coefficients $c_0, \dots, c_{k-1}$ satisfying

$$\mathsf{T}^k\mathbf{v} + c_{k-1}\mathsf{T}^{k-1}\mathbf{v} + \cdots + c_1\mathsf{T}\mathbf{v} + c_0\mathbf{v} = 0.$$

a) (5) Show that $\mathfrak{B} = \{\mathbf{v}, \mathsf{T}\mathbf{v}, \ldots, \mathsf{T}^{k-1}\mathbf{v}\}$ forms a basis for W, the T-cyclic subspace of V generated by $\mathbf{v}$, and $\dim \mathsf{W} = k$.

b) (8) Prove that the characteristic equation of T_W is

$$(-1)^k(x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0).$$

6. (8×2) Let V be an inner product space and W be a finite-dimensional subspace of V.

   a) Prove that for every $\mathbf{v} \in \mathsf{V}$, there exist unique vectors $\mathbf{w} \in \mathsf{W}$, $\mathbf{w}' \in \mathsf{W}^\perp$ satisfying $\mathbf{v} = \mathbf{w} + \mathbf{w}'$.

   b) Find the minimal solution of the equation:

$$\begin{aligned} x + y + 2z &= 6 \\ 2x - z &= 5 \end{aligned}$$

   Explain briefly why your solution is actually minimal.

7. (8) Let T be a linear operator on a finite-dimensional vector space V and W be a subspace of V. Prove that W is T-invariant if and only if W^0 is T^t-invariant, where $\mathsf{W}^0 = \{f \in \mathsf{V}^* : f(\mathbf{v}) = 0 \text{ for } \mathbf{v} \in \mathsf{W}\}$ is the annihilator of W, and $\mathsf{T}^t : \mathsf{V}^* \to \mathsf{V}^*$ is the dual map induced by T.

8. (5×3) Let T and U be linear operators on finite-dimensional vector spaces V and W, respectively.

   a) Prove that there is a unique linear operator on $\mathsf{V} \otimes \mathsf{W}$, say $\mathsf{T} \otimes \mathsf{U}$, satisfying

$$(\mathsf{T} \otimes \mathsf{U})(\mathbf{v} \otimes \mathbf{w}) = \mathsf{T}(\mathbf{v}) \otimes \mathsf{U}(\mathbf{w}) \text{ for } \mathbf{v} \in \mathsf{V}, \mathbf{w} \in \mathsf{W}.$$

   b) Determine the matrix representation of $\mathsf{T} \otimes \mathsf{U}$ with respect to the standard (lexicographically) ordered basis.

   c) Describe $\det(\mathsf{T} \otimes \mathsf{U})$ in terms of $\det \mathsf{T}$ and $\det \mathsf{U}$. You don't have to justify your answer.

9. (5) Let H be a non-degenerate Hermitian form on $F = \mathbb{Q}(i)$ given by the matrix representation

$$\begin{pmatrix} 2 & 1+2i \\ 1-2i & 2 \end{pmatrix}$$

with respect to the standard ordered basis. Find an orthogonal basis for F^2, that is, matrix representation of H with respect to this basis is diagonal.