2014년 2학기 TA 자격 시험: 선형대수학

2014/07/22, 15:00–17:30

1. Consider $\mathsf{V} = \mathbb{F}_3^3$, the 3-dimensional vector space over $\mathbb{F}_3$, the finite field of three elements.

   a) Find the number of *invertible* linear operators on V.

   b) Let B be the bilinear form on V with the Gram matrix with respect to the standard ordred basis $M_{\mathcal{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then find the number of elements of the *orthogonal group*, that is, find the number of the set $\{P \in \mathcal{M}_3(\mathbb{F}_3) : P^t M_{\mathcal{S}} P = M_{\mathcal{S}}\}$.

2. Evaluate the determinants of the following 2014×2014 matrices.

   a) $A = (a_{ij}) = \begin{cases} 2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise} \end{cases}$ b) $B = (b_{ij}) = \begin{cases} a & \text{for } i = j, \\ b & \text{for } i \neq j \end{cases}$

3. Give a brief answer for each problem.

   a) Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Then show that if λ is a nonzero eigenvalue of AB, it is also an eigenvalue of BA.

   b) For any orthogonal operator T on $\mathbb{R}^3$ with determinant 1, prove that $\det(\mathsf{T} - \mathsf{I}) = 0$.

   c) Find a diagonal matrix D and *unitary* matrices $P,\ Q$ satisfying $PDQ = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

4. Let V be an *infinite-dimensional* complex inner product space.

   a) Give an example of a maximal orthonormal set which cannot be a basis for V.

   b) Give an example of *normal* operator T which has no eigenvalue.

5. Let $\mathsf{T} = \mathsf{L}_A$, the linear operator on F^3 defined by left-multiplication of A, where

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -7 & 10 & 15 \\ 5 & -7 & -11 \end{pmatrix}.$$

   For each field $\mathbb{Q}$ and $\mathbb{F}_3$, find a rational canonical form or a Jordan canonical form of A. (If the characteristic polynomial over F splits, give a Jordan form. Otherwise, give a rational form.) You have to find a suitable canonical basis for each case.

6. Let $\mathsf{V} = \mathsf{P}^2(\mathbb{R})$, and $\mathcal{B} = \{1, x\}$ be the standard ordered basis for $\mathsf{W} \leq \mathsf{V}$, and $\langle p(x), q(x)\rangle := \displaystyle\int_0^\infty e^{-x}p(x)q(x)dx$ be an inner product on V.

   a) Construct an orthonormal basis for W with respect to the given inner product.

   b) Decompose $x^2 = p(x) + q(x)$, where $p(x) \in \mathsf{W}$ and $q(x) \in \mathsf{W}^\perp$.

   c) Let $\mathsf{T}(p(x)) = p'(x)$ be a linear operator on V. Then find the matrix representation of the adjoint operator T_W^* with respect to the standard ordered basis for W. (One can easily verify that W is T-invariant and T_W also has the adjoint.)

7. Let B be a *non-degenerate* bilinear form on a finite dimensional space V.

   a) For any linear functional f on V, prove that there exists a unique vector $\mathbf{w} \in \mathsf{V}$ satisfying $f(\mathbf{v}) = B(\mathbf{v}, \mathbf{w})$.

   b) For any subspace $\mathsf{W} \leq \mathsf{V}$, show that $\dim \mathsf{W} + \dim \mathsf{W}^\perp = \dim \mathsf{V}$.

   c) Let B be a symmetric bilinear form on $\mathbb{Q}^3$ with the Gram matrix with respect to the standard ordered basis $M_\mathcal{S} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, and let $\mathsf{W} := \langle \mathbf{e}_1, \mathbf{e}_3 \rangle$. Then find a basis for $\mathsf{W}^\perp$.

   d) If W is a non-singular subspace of V, that is, rad(W)$= \{0\}$, prove that $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}^\perp$.

   e) If B is *alternating*, then show that $\dim \mathsf{V}$ must be *even*.