2014년 1학기 TA 자격 시험: 선형대수학

2014/01/24, 15:00–17:30

In each problem, assume that $\mathsf{V},\mathsf{W},\ldots$ are finite-dimensional vector spaces over a field F.

1. Let $\mathcal{M}_n(F)$ be the set of $n\times n$ matrices over F.
   a) Find the number of *invertible* matrices in $\mathcal{M}_n(F)$ when $F=\mathbb{F}_q$, the finite field of q elements.
   b) Prove that the ring $\mathcal{M}_n(F)$ has no nonzero proper ideal.

2. Let T be a linear operator on V and W be a proper T-invariant subspace of V.
   a) For any $\mathbf{v}\in\mathsf{V}$, show that $I := \{f(x)\in F[x] : f(\mathsf{T})\mathbf{v}\in\mathsf{W}\}$ is an ideal in $F[x]$.
   b) Assume the minimal polynomial of T splits into linear factors. Then prove that there exists a vector $\mathbf{v}\notin\mathsf{W}$ in V, but $(\mathsf{T}-\lambda\mathsf{I}_\mathsf{V})\mathbf{v}\in\mathsf{W}$ for some eigenvalue λ of T. (Hint: Consider the monic generator of I in a).)

3. Let $\mathsf{T}=\mathsf{L}_A$, the linear operator on F^3 defined by $\mathsf{T}\mathbf{v}=A\mathbf{v}$, where

$$A=\begin{pmatrix}8 & 24 & -5\\ 3 & -9 & 2\\ 0 & 1 & 1\end{pmatrix}.$$

   a) When $F=\mathbb{Q}$, find a rational canonical basis for F^3 to find a rational canonical form of A.
   b) When $F=\mathbb{F}_3$, the finite field of 3 elements, find a Jordan canonical basis for F^3 to find a Jordan canonical form of A.

4. Let T be a linear operator on an inner product space V.
   a) Assume the inner product is *complex* and $\langle\mathsf{T}\mathbf{v},\mathbf{v}\rangle=0$ for any $\mathbf{v}\in\mathsf{V}$, then show that $\mathsf{T}=0$.
   b) Assume the inner product is *real* and $||\mathsf{T}\mathbf{v}||=||\mathbf{v}||$ for any $\mathbf{v}\in\mathsf{V}$, then show that $\mathsf{T}^*\mathsf{T}=\mathsf{I}_\mathsf{V}$.

5. Let T be a linear operator on an inner product space V of rank r.

   a) Prove that there exist orthonormal bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for V and positive scalars $\sigma_1 \geq \cdots \geq \sigma_r$ such that

$$\mathsf{T}\mathbf{v}_i = \begin{cases} \sigma_i \mathbf{w}_i & \text{for } 1 \leq i \leq r, \\ 0 & \text{for } i > r. \end{cases}$$

   b) Let $\mathsf{T} = \mathsf{L}_A$, the linear operator on $\mathbb{R}^3$ defined by $\mathsf{T}\mathbf{v} = A\mathbf{v}$, where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix}.$$

   Find the least square approximation of the equation $A\mathbf{x} = \mathbf{b} = (-1, 2, 1)^t$, that is, find $\mathbf{x} \in \mathbb{R}^3$ minimizing $||A\mathbf{x} - \mathbf{b}||^2$.

6. Let B be a non-degenerate symmetric bilinear form on V. Let $\mathcal{B} = \{\mathbf{v}_i\}$ be a basis for V and $\mathcal{B}^* = \{f_i\}$ be the dual basis of $\mathcal{B}$.

   a) Consider $\varphi : \mathsf{V} \to \mathsf{V}^*$ defined by $\varphi(\mathbf{w}) = f_{\mathbf{w}}$, where $f_{\mathbf{w}}(\mathbf{v}) = B(\mathbf{v}, \mathbf{w})$. Show that φ is an isomorphism and find the matrix representation $[\varphi]_{\mathcal{B},\mathcal{B}^*}$.

   b) Show that there exists another basis $\mathcal{C} = \{\mathbf{w}_j\}$ for V satisfying $B(\mathbf{v}_i, \mathbf{w}_j) = \delta_{ij}$, and find the matrix representation of B with respect to $\mathcal{C}$.

   c) Let $M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ be the matrix representation of B on $\mathsf{V} = \mathbb{Q}^3$ with respect to the standard ordered basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then construct an orthogonal basis for V.

7. a) State the universal property of the tensor product of R-modules M, N.

   b) Let $\mathcal{B} = \{\mathbf{v}_i\}$, $\mathcal{C} = \{\mathbf{w}_j\}$ be bases for V,W respectively. Then show that $\{\mathbf{v}_i \otimes \mathbf{w}_j\}$ forms a basis for $\mathsf{V} \otimes \mathsf{W}$.

   c) Let $f \in \mathsf{V}^*$, $g \in \mathsf{W}^*$ be linear functionals on V,W respectively. Consider the *bilinear* map $\varphi : \mathsf{V} \times \mathsf{W} \to F$ defined by $\varphi(\mathbf{v}, \mathbf{w}) = f(\mathbf{v})g(\mathbf{w})$. Then use the universal property in a) to induce a map $\Phi : \mathsf{V}^* \otimes \mathsf{W}^* \to (\mathsf{V} \otimes \mathsf{W})^*$ defined by $\Phi(f \otimes g)(\mathbf{v} \otimes \mathbf{w}) = f(\mathbf{v})g(\mathbf{w})$, and show that Φ is an isomorphism.