2015년 1학기 TA 자격 시험: 선형대수학

2015/01/23, 14:30–17:00

1. State the following theorems. You don't have to give any proof.

   a) Primary decompsition theorem for finitely generated torsion modules over a principal ideal domain.

   b) Cyclic decomposition theorem for finitely generated primary modules over a principal ideal domain.

2. Give a brief proof for each statement.

   a) The matrix ring $\mathcal{M}_n(F)$ has no nontrivial proper ideal.

   b) Let $f(t)$ be a polynomial of degree n with the leading coefficient $(-1)^n$. Then there exists an $n \times n$ matrix whose characteristic polynomial is $f(t)$.

   c) Let M be a *free* module of rank 2 over a principal ideal domain R, and N be its submodule. Then N is also free.

   d) Let T be an linear operator on a finite-dimensional complex inner product space V, and T^* be its adjoint. Then $\mathsf{T}^* = g(\mathsf{T})$ for some polynomial $g(t) \in \mathbb{C}[t]$ if and only if T is normal.

3. Consider $\mathsf{T} = \mathsf{L}_A$, the linear operator on $\mathbb{R}^{2015}$, where

$$\begin{pmatrix} 1 & 2 & \cdots & 2015 \\ 2016 & 2017 & \cdots & 4030 \\ \vdots & \vdots & & \vdots \\ (2015)^2 - 2014 & (2015)^2 - 2013 & \cdots & (2015)^2 \end{pmatrix}.$$

   a) (10) Show that the 2-dimensional subspace generated by $\{(1, 1, \ldots, 1)^t, (1, 2, \ldots, 2015)^t\}$ is T-invariant.

   b) Determine $\chi_{\mathsf{T}}(t)$, the characteristic polynomial of T.

   c) (10) Determine $m_{\mathsf{T}}(t)$, the minimal polynomial of T.

4. Let V be a complex inner product space and W be a subspace of V.

   a) Show that $(\mathsf{W}^{\perp})^{\perp} = \mathsf{W}$ if W is *finite-dimensional.*

   b) Give an example of a subspace W such that the result of a) does not hold when W is *infinite-dimensional.*

5. Let T be a linear operator on an inner product space V of rank r.

a) Prove that there exist orthonormal bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for V and positive scalars $\sigma_1 \geq \cdots \geq \sigma_r$ such that

$$\mathsf{T}\mathbf{v}_i = \begin{cases} \sigma_i \mathbf{w}_i & \text{for } 1 \leq i \leq r, \\ 0 & \text{for } i > r. \end{cases}$$

b) (10) Let $\mathsf{T} = \mathsf{L}_A$, the linear operator on $\mathbb{R}^3$, where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix}.$$

Then find the *pseudoinverse* of A, that is, find the matrix representation of the map $\mathsf{T}^- : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $\mathsf{T}^-\mathsf{T} = \mathsf{Id}_{\mathsf{N(T)}^\perp}$ and $\mathsf{T}\mathsf{T}^- = \mathsf{Id}_{\mathsf{R(T)}}$ with respect to the standard ordered basis.

c) Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, which is *inconsistent*. Then show that $\mathbf{x} = A^-\mathbf{b}$ is the least square approximation of this system minimizing $||A\mathbf{x} - \mathbf{b}||$.

6. Let H be a *non-degenerate* Hermitian form on a finite dimensional space V over $\mathbb{Q}(\sqrt{-2})$.

a) For any linear functional f on V, prove that there exists a unique vector $\mathbf{w} \in \mathsf{V}$ satisfying $f(\mathbf{v}) = H(\mathbf{v}, \mathbf{w})$.

b) For any subspace $\mathsf{W} \leq \mathsf{V}$, show that $\dim \mathsf{W} + \dim \mathsf{W}^\perp = \dim \mathsf{V}$.

c) Let H be a Hermitian form on V with the matrix representation with respect to the standard ordered basis $M_{\mathcal{S}} = \begin{pmatrix} 3 & \sqrt{-2} \\ -\sqrt{-2} & 3 \end{pmatrix}$. Then construct an orthogonal basis, that is, find a basis $\mathcal{B}$ such that $M_{\mathcal{B}}$ is diagonal.