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The representation of quadratic forms by almost universal forms of higher rank

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Abstract In this article, we prove that there are only finitely many positive definite integral quadratic forms of rank $n+3$ ($n \geq 2$) that represent all positive definite integral quadratic forms of rank n but finitely many exceptions. Furthermore we determine all diagonal quadratic forms having such property and its exceptions remaining four as candidates.

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1 Introduction

After the famous Lagrange's four square theorem [15], all positive definite classic integral quaternary quadratic forms that represent all positive integers, which we call *universal quaternary forms,* have been completely determined (see [1],[3],[5],[23] and [24]). In 1926, Kloosterman [14] determined all positive definite diagonal quaternary quadratic forms that represent all sufficiently large integers, which we call *almost universal forms,* although he did not succeed in proving the almost universality of four candidate forms. Pall [21] proved the almost universality for the remaining quadratic forms and in fact, there are exactly 199 almost universal quaternary diagonal quadratic forms that are anisotropic over some ring of p-adic integers. Furthermore Pall and Ross [22] proved that there exist only finitely many almost universal quaternary quadratic forms that are anisotropic over some ring of p-adic integers by providing a upper bound of the discriminant of such forms. On the other hand, they proved that every positive definite quaternary quadratic form L such that $L_p := L \otimes \mathbb{Z}_p$ represents all p-adic integers and is isotropic over $\mathbb{Z}_p$ for all primes p is almost universal (see also

Theorem 2.1 of [8]). Therefore there are infinitely many almost universal quaternary quadratic forms.

As a natural generalization to higher rank case, we [10] proved that there are exactly eleven quinary positive integral quadratic forms that represent all positive integral binary quadratic forms. (See also [11], [12] and [18].) As a natural generalization of a result of Halmos [7], Hwang [9] proved that there are exactly 3 quinary diagonal positive definite integral quadratic forms that represent all binary positive definite integral quadratic forms except only one.

In this paper, we prove that if $n \geq 2$, there are only finitely many positive definite integral quadratic forms of rank $n+3$ that represent all but at most finitely many equivalence classes of positive definite integral quadratic forms of rank n. We call such quadratic forms *almost n-universal* quadratic forms. Furthermore we determine all candidates for almost n-universal diagonal quadratic forms of rank $n+3$ and prove the almost n-universality, and determine the n-ary lattices that are not represented, for all but four of the candidate forms.

We shall adopt lattice theoretic language. A *$\mathbb{Z}$-lattice* L is a finitely generated free $\mathbb{Z}$-module in $\mathbb{R}^n$ equipped with a non-degenerate symmetric bilinear form B, such that $B(L, L) \subseteq \mathbb{Z}$. The corresponding quadratic map is denoted by Q.

For a $\mathbb{Z}$-lattice $L = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2 + \cdots + \mathbb{Z}\mathbf{e}_n$ with basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$, we write

$$L = (B(\mathbf{e}_i, \mathbf{e}_j)).$$

By $L = L_1 \perp L_2$ we mean $L = L_1 \oplus L_2$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$ for all $\mathbf{e}_1 \in L_1, \mathbf{e}_2 \in L_2$. We call L *diagonal* if it admits an orthogonal basis and in this case, we simply write

$$L = \langle Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n) \rangle,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is an orthogonal basis of L. We call L *non-diagonal* otherwise. L is called *positive definite* or simply *positive* if $Q(\mathbf{e}) > 0$ for any $\mathbf{e} \in L, \mathbf{e} \neq \mathbf{0}$. As usual, $dL := \det(B(\mathbf{e}_i, \mathbf{e}_j))$ is called the *discriminant* of L. For a $\mathbb{Z}$-lattice L and a prime p, we define $L_p := \mathbb{Z}_p L$ and call it the localization of L at p.

Let ℓ, L be $\mathbb{Z}$-lattices. We say L *represents* ℓ if there is an injective linear map from ℓ into L that preserves the bilinear form, and write $\ell \to L$. Such a map will be called a *representation*. A representation is called an *isometry* if it is surjective. Furthermore we say ℓ is *primitively* represented by L if there exists an isometry σ from ℓ to L such that $\sigma(\ell)$ is a primitive sublattice of L. We say two $\mathbb{Z}$-lattices L, K are isometric if there is an isometry between them, and write $L \cong K$. The set of all $\mathbb{Z}$-lattices that are isometric to L is called the *class* of L, denoted by $\mathrm{cls}(L)$. We define $\ell_p \to L_p$ and $L_p \cong K_p$ in a similar manner over $\mathbb{Z}_p$. The set of all $\mathbb{Z}$-lattices K such that $L_p \cong K_p$ for all prime spots p (including ∞) is called the *genus* of L, denoted by $\mathrm{gen}(L)$. The number of classes in a genus is called the *class number* of the genus (or of any $\mathbb{Z}$-lattice in the genus), which is known to be finite. For the class number of each $\mathbb{Z}$-lattice, see [4], [16] and [17].

A positive $\mathbb{Z}$-lattice L is called *almost n-universal* if L represents all n-ary positive $\mathbb{Z}$-lattices except those in only finitely many equivalence classes. The notion of (*locally*) *n-universal* is defined in a similar manner. If a $\mathbb{Z}$-lattice L is almost n-universal, then the rank of L is greater than or equal to $n+3$.

We set

$$[a,b,c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

for convenience. For unexplained terminologies, notations, and basic facts about $\mathbb{Z}$-lattices, we refer the readers to O'Meara's book [19].

2 Finiteness of almost n-universal $\mathbb{Z}$-lattices of rank $n+2 (n \geq 2)$

The following definition of *successive minimum* is adapted from [[2], Chapter 12].

Definition 2.1 *Let L be a $\mathbb{Z}$-lattice of rank n. We define the j-th successive minimum $m_j(L)$ of L to be the positive integer such that*
(1) the set of vectors $v \in L$ with $Q(v) \leq m_j(L)$ spans a subspace of dimension greater than or equal to j ;
(2) the set of vectors $v \in L$ with $Q(v) < m_j(L)$ spans a subspace of dimension less than j.

It is clear that $m_1(L) \leq m_2(L) \leq \cdots \leq m_n(L)$ and there is a set of linearly independent vectors x_j, $j = 1, 2, \ldots, n$, such that $Q(x_j) = m_j(L)$. If dL is the discriminant of L, then

$$dL \leq \prod_{i=1}^{n} m_i(L) \leq C \cdot dL, \tag{1}$$

for a constant C depending only on n (see [6]).

Proposition 2.2 *If there exist only finitely many almost 2-universal quinary $\mathbb{Z}$-lattices, then there exist only finitely many almost n-universal $\mathbb{Z}$-lattices of rank $n+3$ for $n \geq 2$.*

Proof. Assume that L is an almost n-universal $\mathbb{Z}$-lattice of rank $n+3$. If L cannot represent a $\mathbb{Z}$-lattice ℓ of rank $n-1$, then for any positive integer a, $\ell \perp \langle a \rangle \not\longrightarrow L$. Therefore L must be $(n-1)$-universal $\mathbb{Z}$-lattice. This implies that $n \leq 6$ by [18]. Furthermore since L represents 1, $L \simeq L' \perp \langle 1 \rangle$ for an almost $(n-1)$-universal $\mathbb{Z}$-lattice L'. Therefore the desired result follows.

Now we prove that there exist only finitely many almost 2-universal quinary $\mathbb{Z}$-lattices. Note that every almost 2-universal $\mathbb{Z}$-lattice is, in fact, locally 2-universal.

Lemma 2.3 *Let L be a quinary $\mathbb{Z}_p$-lattice and $L = \perp L_i$ be it's Jordan decomposition. We define $d_i := d(L_i)$ the discriminant of L_i and r_i the rank of L_i. Then L is 2-universal over $\mathbb{Z}_p$ if and only if*

(1) $p \neq 2$

$$r_0 = 5 \quad or \quad r_0 = 4,\ d_0 = 1 \quad or$$
$$r_0 = 4,\ d_0 = \Delta_p \ and\ r_1 = 1 \ or \ \ r_0 = 3,\ r_1 = 2.$$

(2) $p = 2$

$$r_0 = 5 \quad or \quad r_0 = 4,\ r_1 = 1 \quad or \quad r_0 = 4,\ d_0 \equiv 3 \pmod 4 \ and\ r_2 = 1 \ or$$
$$r_0 = 3,\ 1 \leq r_1 \ and\ r_1 + r_2 + r_3 = 2.$$

Here Δ_p is any nonsquare unit in $\mathbb{Z}_p$.

Proof. This follows directly from [20].

Lemma 2.4 *Let L be any locally 2-universal $\mathbb{Z}$-lattice of rank 5 and for all prime p, let $d(L_p) = p^{u_p}\alpha_p$, where α_p is a unit in $\mathbb{Z}_p$ and u_p is a non-negative integer. Then there exists a prime p dividing $2dL$ such that L cannot primitively represent binary $\mathbb{Z}$-lattices ℓ for which*

$$\ell_p \simeq \langle p^{\epsilon_p}\alpha_p, p^k\beta_p\rangle,$$

where ϵ_p is 0 or 1, respectively the parity of u_p, β_p is any unit in $\mathbb{Z}_p$ and $k \geq 2$ if p is odd and $k \geq 7$ otherwise.

Proof. As a quadratic space, $\mathbb{Q}L := \mathbb{Q} \otimes L$ can be decomposed by $\mathbb{Q}L \simeq \langle dL\rangle \perp V$ for a quadratic space V with $dV = 1$. By the reciprocity law for the Hasse symbol,

$$1 = \prod_p S_p(\mathbb{Q}L) = \prod_p (dL, dL)_p \cdot S_p(V) = \prod_p S_p(V).$$

Therefore for at least one p, which we will call *a core prime of L*, V_p is anisotropic by [[19],63.17]. Assume that p is odd. If $\epsilon_p = 0$, then $L_p \simeq \langle 1, -\Delta_p, \alpha_p, p, -\Delta_p p\rangle$ by Lemma 2.3. Therefore the desired result follows. If $\epsilon_p = 1$, then $L_p \simeq \langle 1, 1, 1, \Delta_p, \alpha_p\Delta_p p\rangle$. Therefore the desired result follows from Lemma 2.1 of [13]. Now assume that $p = 2$. Note that the $\mathbb{Z}_2$-lattice $K := \langle 1, 1, 1, 1\rangle$ cannot primitively represent all p-adic integers divisible by 8. Hence any sublattice of K with index 2^d cannot primitively represent all integers divisible by 2^{2d+3}. Let L' be the orthogonal complement of $\langle 2^{\epsilon_2}\alpha_2\rangle$ in L_2. Then clearly, L' is a sublattice of K with index 2^n, where $n = 0, 1, 2$. Therefore the desired result follows.

Remark 2.5 *If $p = 2$, the minimum possible value of k can be smaller than 7, depending on L.*

Definition 2.6 *Let L be a $\mathbb{Z}$-lattice. A $\mathbb{Z}$-lattice ℓ is called a core $\mathbb{Z}$-lattice of L if the failure of L to represent ℓ implies that L fails to represent infinitely many $\mathbb{Z}$-sublattices of ℓ.*

If L is almost n-universal, then L must represent all n-ary core $\mathbb{Z}$-lattices of L.

Lemma 2.7 *Let L be a locally 2-universal quinary $\mathbb{Z}$-lattice. Then L always has a binary core $\mathbb{Z}$-lattice.*

Proof. Under the same notations of Lemma 2.4, let $\ell = \mathbb{Z}x + \mathbb{Z}y$ be a binary $\mathbb{Z}$-lattice satisfying all conditions given there. Furthermore we may assume that the matrix generated by the vectors x and y is sufficiently close to the form appearing in Lemma 2.4 over $\mathbb{Z}_p$. For a positive integer n, assume that $\ell(n) := \mathbb{Z}x + \mathbb{Z}(p^n y)$ is represented by L. Let σ be it's representation. Since $\ell(n)$ is not primitively represented by L, there exist integers a, b satisfying $\gcd(a, b, p) = 1$ such that $a\sigma(x) + b\sigma(p^n y) = pz$ for a vector $z \in L$. Hence $a \equiv 0 \pmod{p}$ and $\sigma(p^n y)$ is not a primitive vector in L. Therefore $\ell(n-1)$ is represented by L. From this follows the lemma.

Theorem 2.8 *There exist only finitely many almost 2-universal $\mathbb{Z}$-lattices of rank 5.*

Proof. Let L be a locally 2-universal quinary $\mathbb{Z}$-lattice. We prove that if the 5-th successive minimum of L is sufficiently large, then L cannot represent infinitely many binary $\mathbb{Z}$-lattices. This implies the desired result by (2.1). Assume that $m_5(L)$ is sufficiently large. Let p be a core prime of L defined on Lemma 2.4.

Since L is 1-universal and $m_5(L)$ is sufficiently large, the primitive quaternary sublattice, say L', containing x_1, x_2, x_3, x_4 such that $Q(x_i) = m_i(L)$ must be 1-universal (see [3]). Note that L' is isometric to one of the quaternary 1-universal $\mathbb{Z}$-lattices, which exist only finitely many. Therefore by (2.1), there exist real numbers M, N independent of L such that $MdL \le m_5(L) \le NdL$.

First assume that p is an odd prime. Since $(L')_2$ is not 2-universal (see [10]), there exists a primitive binary $\mathbb{Z}_2$-lattice K (i.e., there does not exist a binary $\mathbb{Z}_2$-lattice properly containing K), such that $K \not\longrightarrow L_2$. Note that either K represents a unit, say η, or is isometric to $[2, 1, 2]$. Let q be an integer in $\{3, 11, 5, 13, 7, 23, 17, 41, 6, 22, 10, 26, 14, 46, 34, 82\}$ such that $pq \in d(K)(\mathbb{Z}_2^*)^2$ and $\gcd(p, q) = 1$.

Assume that $\epsilon_p = 0$. Since $L_p \simeq \langle 1, -\Delta, \alpha_p, p, -\Delta_p p\rangle$, L'_p is not unimodular. Therefore p is bounded by some constant C because of the finiteness of L'. Let a be a positive integer such that $a < 8p$ and

$$\begin{array}{ll} a \equiv \alpha_p \pmod{p} \text{ and } a \equiv \eta \pmod{8} & \text{if } \langle \eta \rangle \longrightarrow K, \\ a \equiv 2\alpha_p \pmod{p} \text{ and } a \equiv 1 \pmod{2} & \text{if } K \simeq [2, 1, 2]\ . \end{array}$$

If $8C^2 q \le m_5(L)$, then $[a, 0, apq]$ or $[2, 1, 2a]$, respectively to the condition of K, is the core $\mathbb{Z}$-lattice of L that is not represented by L.

Now assume that $\epsilon_p = 1$. Let a be a positive integer such that $a < 8p$ and

$$\begin{array}{ll} a \equiv \alpha_p \pmod{p} \text{ and } a \equiv \eta p \pmod{8} & \text{if } \langle \eta \rangle \longrightarrow K, \\ a \equiv 2\alpha_p \pmod{p} \text{ and } a \equiv 1 \pmod{2} & \text{if } K \simeq [2, 1, 2]\ . \end{array}$$

If $16p\max(p,q) < MdL$, then $[ap, 0, aq]$ or $[2p, p, 2ap]$ is the core $\mathbb{Z}$-lattice of L that is not represented by L. If $MdL \le 16p\max(p,q)$, then $m_5(L) \le \frac{16Np\max(p,q)}{M}$. Hence we may assume that p is sufficiently large. Since both cases can be done in a similar manner, we only provide a proof of the case when $\langle \eta \rangle \longrightarrow K$.

Let Ω be the product of all primes not greater than $\frac{(q+1)}{M}$. Take an integer a satisfying

$$\left(\frac{a}{p}\right) = \left(\frac{\alpha_p q}{p}\right), \quad a \equiv \eta \pmod 8, \quad \left(\frac{pq}{a}\right) = \left(\frac{-1}{a}\right) \quad \text{and} \quad \gcd(a, \Omega) = 1.$$

Here $(-)$ is the Jacobi symbol. Furthermore we can choose a such that $a < Cp^{\frac{3}{8}+\epsilon}$ for a constant C and $\epsilon > 0$ by Corollary 3.3 of [6]. We assume that p is sufficiently large so that $a \le p^{\frac{1}{2}} \le Mp$. We let $\ell = [a, b, c]$, where b, c is positive integers such that $0 < b < a$ and $pq = -b^2 + ac$. Note that such integers always exist by the above conditions. Since $a, c < Mp \le m_5(L)$ and $\ell_2 \simeq K$, ℓ is the core $\mathbb{Z}$-lattice of L that is not represented by L.

Lastly, assume that $p = 2$. If $d(L')$ is a square integer, then there exists a bounded prime r such that L'_r is anisotropic by a similar reasoning to Lemma 2.4. If r is odd and $r^2 < m_5(L)$, then for all positive integers s, $[r^2, 1, s]$ is not represented by L, because r^2 cannot be primitively represented by L. If $r = 2$ and $64 < m_5(L)$, then $[64, 1, s]$ cannot represented by L by a similar reasoning to above and by Lemma 3 of [21]. Assume that $d(L')$ is not a square integer and let r be a bounded odd prime such that $(L')_r$ is not universal. The existence of such a bounded prime r follows from the finiteness of L' up to isometry. Let M be a fixed binary $\mathbb{Z}_r$-lattice such that $M \not\longrightarrow (L')_r$. Then there exists a $\mathbb{Z}$-lattice ℓ such that $\ell_r \simeq M$ and ℓ is isometric to the $\mathbb{Z}_2$-lattice given by Lemma 2.4 for fixed k. For all possible ℓ's, if $m_2(\ell) < m_5(L)$, then ℓ is a core $\mathbb{Z}$-lattice that is not represented by L. Therefore the theorem follows.

3 Almost n-universal diagonal $\mathbb{Z}$-lattices of rank $n+3$ $(n \ge 2)$

In this section, we determine all candidates of almost n-universal diagonal $\mathbb{Z}$-lattices of rank $n+3$ for $n \ge 2$. Let $L = \langle a, b, c, d, e \rangle$ be a locally 2-universal $\mathbb{Z}$-lattice. We assume that $0 < a \le b \le c \le d \le e$. To represent binary $\mathbb{Z}$-lattices of the form $[1, 0, s_1], [2, 1, s_2]$ and $[3, 1, s_3]$, $a = b = 1$ and $c = 1$ or 2. If $c = 1$ and $d \ge 4$, then $[4, 1, s_4] \not\longrightarrow L$. If $c = 2$ and $d \ge 6$ then $[6, 1, s_5] \not\longrightarrow L$. Therefore L contains one of the following quaternary $\mathbb{Z}$-lattices:

$$\langle 1,1,1,1 \rangle, \quad \langle 1,1,1,2 \rangle, \quad \langle 1,1,1,3 \rangle,$$
$$\langle 1,1,2,2 \rangle, \quad \langle 1,1,2,3 \rangle, \quad \langle 1,1,2,4 \rangle, \quad \langle 1,1,2,5 \rangle.$$

For each quinary $\mathbb{Z}$-lattice L, if L is not almost 2-universal, we will give a binary core $\mathbb{Z}$-lattice that is not represented by L for most cases. We call such a $\mathbb{Z}$-lattice *an exceptional core* $\mathbb{Z}$*-lattice of* L. When 2 is a core prime of L and ℓ is an exceptional core $\mathbb{Z}$-lattice of L, then we always write $\ell = [a, b, c]$ if $[a, b \cdot 2^m, c \cdot 2^{2m}]$ is not represented by L. For a given odd core prime p and an integer t, we write $t \sim 1$ if t is a square in $\mathbb{Z}_p^*$ and we write $t \sim \Delta$, if t is a nonsquare unit in $\mathbb{Z}_p^*$. If we write $[a, b, c] \simeq [a', b', c'] \longrightarrow L_p$, then it means $[a, b, c]$ is isometric to $[a', b', c']$ over $\mathbb{Z}_p$ and is represented by L over $\mathbb{Z}_p$, i.e., whether $[a, b, c]$ is defined on $\mathbb{Z}$ or $\mathbb{Z}_p$, it depends on the notation of the right lattice.

Case 1 $L(e) := L' \perp \langle e \rangle = \langle 1, 1, 1, 1, e \rangle$

If $e = 1, 2, 3$, then $L(e)$ has class number 1 and is locally 2-universal. Therefore $L(e)$ is 2-universal $\mathbb{Z}$-lattice by [[19], 102.5]. If e is even greater than 3 then $[4, 1, 2s]$ is not represented by $L(e)$ for all $s \geq 2$. If $e \geq 8$ then $[8, 1, s]$ is not represented by $L(e)$ for all $s \geq 8$. Clearly $[3, 0, 5] \simeq [23, 10, 5]$ is not represented by $L(7)$. We show that this binary $\mathbb{Z}$-lattice is an exceptional core $\mathbb{Z}$-lattice of $L(7)$. Assume that $\ell(m) := [23, 10 \cdot 2^m, 5 \cdot 2^{2m}] \longrightarrow L(7)$. Then

$$\begin{aligned} \ell(s,t,m) : &= [23 - 7s^2, 10 \cdot 2^m - 7st, 5 \cdot 2^{2m} - 7t^2] \\ &= \begin{pmatrix} 23 - 7s^2 & 10 \cdot 2^m - 7st \\ 10 \cdot 2^m - 7st & 5 \cdot 2^{2m} - 7t^2 \end{pmatrix} \longrightarrow L', \end{aligned}$$

for some integers s, t. Note that $s = 0$ or 1 and if $s = 1$ then $t \neq 0$ by the positive definiteness of $\ell(s, t, m)$. For all possible s, t, if we calculate the discriminant of $\ell(s, t, m)$, we can easily check that $\mathbb{Q}_2\ell(s, t, m)$ is always a hyperbolic space. This is a contradiction. Therefore $L(7)$ is not almost 2-universal. In Theorem 3.1, we will prove that $L(5)$ is, in fact, almost 2-universal.

Case 2 $L(e) := L' \perp \langle e \rangle = \langle 1, 1, 2, 2, e \rangle$

If $L(e)$ is almost 2-universal, then $\langle 1, 1, 1, 1, e \rangle$ is also almost 2-universal. Therefore, by Case (1), it suffices to check only the cases when $e = 3, 5$. Note that $[2, 1, 2]$ is the only one exception of $L(3)$ (see [9]). In Theorem 3.1, we will prove that $L(5)$ is almost 2-universal.

Case 3 $L(e) := L' \perp \langle e \rangle = \langle 1, 1, 1, 2, e \rangle$

Note that $[2, 0, 7] \simeq [1, 0, 14] \not\longrightarrow (L')_2$. We first consider the case when 2 is a core prime of $L(e)$, i.e., $e = 2^k(8n \pm 1)$, where $k = 1, 2, 3$. Note that if $k \geq 4$, then $L(e)$ is not locally 2-universal by Lemma 2.3. If $e \equiv 1 \pmod 8$ and $e \geq 28$, then $[2, 0, 28]$ is an exceptional core $\mathbb{Z}$-lattice. If $e = 9$ or 25, then $L(e)$ is not locally 2-universal. If $e = 17$, then $[18, 4, 4]$ is an exceptional core $\mathbb{Z}$-lattice of $L(17)$. If $e \equiv 7 \pmod 8$ and $e \geq 15$, then $[14, 0, 4]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$. In Theorem 3.2, we will prove that $L(7)$ is an almost 2-universal $\mathbb{Z}$-lattice. If $e = 2(8n + 1)$, and $e \geq 57$, then $[4, 0, 56]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$. Note that $L(2)$ is a 2-universal $\mathbb{Z}$-lattice with class number 1 and $[9, 1, 25]$ is an exceptional core $\mathbb{Z}$-lattice of $L(34)$. If $e = 2(8n - 1)$, then $[7, 0, 8]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$. If $e = 4(8n + 1)$ and $e \geq 113$, then $[2, 0, 112]$ is an exceptional core

$\mathbb{Z}$-lattice. $[18,2,50]$ is an exceptional core $\mathbb{Z}$-lattice of $L(68)$ and $[3,0,3]$ is the only one exception of $L(4)$ (see [9]). If $e=4(8n-1)$, then $[14,0,16]$ is an exceptional core $\mathbb{Z}$-lattice. If e is divisible by 8, then $L(e)$ cannot represent all binary $\mathbb{Z}$-lattices of the form $[2,0,8a+7]$.

Now we assume that 2 is not a core prime of $L(e)$. Since there exists at least one core prime of $L(e)$, we can find an odd core prime p of $L(e)$. Note that $p \equiv \pm 3 \pmod 8$ and e is divisible by p. Let $e=pt$.

(3.1) $p \equiv 11 \pmod{24}$. In this case, $[2p,p,2p]$ is always an exceptional core $\mathbb{Z}$-lattice of $L(e)$.

(3.2) $p \equiv 19 \pmod{24}$. If $t \sim 1$, then $[12,3,\frac{3(p+1)}{4}]$ is an exceptional core $\mathbb{Z}$-lattice and if $t \sim \Delta$, $[p,0,p]$ is an exceptional core $\mathbb{Z}$-lattice.

(3.3) $p \equiv 5 \pmod{24}$. Assume that $t \sim 1$. If $t \ge 4$, $[6,0,3p]$ is an exceptional core $\mathbb{Z}$-lattice. For the remaining case, since $2 \sim 3 \sim \Delta$, we may assume that $t=1$. If $e \ge 132$, then $[132,33,\frac{3(p+11)}{4}]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$. In the following, the right binary $\mathbb{Z}$-lattice is an exceptional core $\mathbb{Z}$-lattice of the left $\mathbb{Z}$-lattice.

$$(*) \qquad L(101)\ :\ [15,3,41],\quad L(53)\ :\ [14,2,23],\quad L(29)\ :\ [15,6,14].$$

Note that [3,0,3] is the only one exception of $L(5)$ (see [9]). Assume that $t \sim \Delta$. If $t \ge 3$, then $[p,0,2p]$ is an exceptional core $\mathbb{Z}$-lattice. So we may assume that $t=2$. If $e \ne 10$, then $[30,15,\frac{3(p+5)}{2}]$ is an exceptional core $\mathbb{Z}$-lattice. Note that $[10,5,10]$ is an exceptional core $\mathbb{Z}$-lattice of $L(10)$.

(3.4) $p \equiv 13 \pmod{24}$. Assume that $t \sim 1$. If $e \ge 456$, then $[456,57,\frac{3(p+19)}{8}]$ is an exceptional core $\mathbb{Z}$-lattice and if $t \ge 2$, $[6,3,\frac{3(p+1)}{2}]$ is also an exceptional core $\mathbb{Z}$-lattice. Therefore it remains only the cases when $e=13, 37, 61$, 109, 157, 181, 229, 277, 349, 373, 397, 421. If $p \equiv 9 \pmod{10}$, $[30,3,\frac{3(p+1)}{10}]$ is an exceptional core $\mathbb{Z}$-lattice and if $p \equiv 1 \pmod{10}$, $[30,9,\frac{3(p+9)}{10}]$ is an exceptional core $\mathbb{Z}$-lattice. For the other cases, similarly to $(*)$, we have:

$$L[397]\ :\ [21,9,174],\quad L[373]\ :\ [33,3,102],\quad L[277]\ :\ [33,9,78],$$
$$L[157]\ :\ [21,6,69],\quad L[37]\ :\ [7,3,33],\quad L[13]\ :\ [52,13,52].$$

Assume that $t \sim \Delta$. Then, similarly to the subcase (3.3), we may assume that $t=2$. If $p \ge 42$, $[84,21,\frac{3(p+7)}{4}]$ is an exceptional core $\mathbb{Z}$-lattice. Note that $[14,4,17]$ is an exceptional core $\mathbb{Z}$-lattice of $L(74)$ and $[13,0,91]$ is an exceptional core $\mathbb{Z}$-lattice of $L(26)$.

(3.5) $p=3$. If $t \ge 2$, $[3,0,3]$ is an exceptional core $\mathbb{Z}$-lattice. $L(3)$ is a 2-universal $\mathbb{Z}$-lattice with class number 1.

Case 4 $L(e) := L' \perp \langle e\rangle = \langle 1,1,2,4,e\rangle$

Since $L(e)$ is a sublattice of $\langle 1,1,1,2,e\rangle$, it suffices to check the cases when $e=4,5,7$. Note that $L(4)$ is not locally 2-universal. $[4,1,4]$ is an exceptional core $\mathbb{Z}$-lattice of $L(5)$ and $[14,7,14]$ is an exceptional core $\mathbb{Z}$-lattice of $L(7)$.

Case 5 $L(e) := L' \perp \langle e\rangle = \langle 1,1,1,3,e\rangle$

Note that $[3,0,7] \not\longrightarrow (L')_2$ and $[1,0,6] \not\longrightarrow (L')_3$. First, we consider the case when 3 is a core prime of $L(e)$, i.e., $e \equiv 0, 2 \pmod 3$. If $e \equiv 2 \pmod 3$ and $e \geq 8$, $[1,0,6]$ is an exceptional core $\mathbb{Z}$-lattice and $[4,1,4] \simeq [6,3,4]$ is an exceptional core $\mathbb{Z}$-lattice of $L(5)$. Note that $[6, 3^{m+1}, 4\cdot 3^{2m}] \not\longrightarrow L(5)$. If $e \equiv 0 \pmod 3$ and $e \geq 6$, then $[6,3,3a+1] \not\longrightarrow L(e)$. Note that $[2,1,3] \simeq [7,4,3]$ is an exceptional core $\mathbb{Z}$-lattice of $L(3)$, i.e., $[7, 4\cdot 3^m, 3^{2m+1}] \not\longrightarrow L(3)$. Now we always assume that $e \equiv 1 \pmod 3$.

Assume that 2 is a core prime of $L(e)$, i.e., e is one of the following forms $4n+1$, $4(4n+1)$ or $2(4n+3)$ for a non-negative integer n. If $e \equiv 1 \pmod 8$, then $[3,0,28]$ is an exceptional core $\mathbb{Z}$-lattice and if $e \equiv 5 \pmod 8$, $[7,0,12]$ is an exceptional core $\mathbb{Z}$-lattice. If $e = 2(8n+3)$ and $n \geq 2$, $[2,0,40]$ is an exceptional core $\mathbb{Z}$-lattice and $[18,2,18]$ is an exceptional core $\mathbb{Z}$-lattice of $L(22)$. If $e = 2(8n+7)$, then $[10,0,8]$ is an exceptional core $\mathbb{Z}$-lattice. If $e = 4(4n+1)$, then $[2,1,4a+3] \not\longrightarrow L(e)$ for any non-negative integer a.

If 2 is not a core prime of $L(e)$, then there exists a core prime p dividing e such that $p \equiv \pm 5 \pmod{12}$. We let $e = pt$. We assume that $t \sim 1$ in (5.1) $\sim$ (5.3).

(5.1) $p \equiv 5 \pmod{12}$. If $p \equiv 5 \pmod 8$, $[3, 1, \frac{p+1}{3}]$ is an exceptional core $\mathbb{Z}$-lattice and if $p \equiv 1 \pmod 8$ and $e \geq 39$, $[39, 13, \frac{p+13}{3}]$ is an exceptional core $\mathbb{Z}$-lattice. Lastly, $[34,17,34]$ is an exceptional core $\mathbb{Z}$-lattice of $L(34)$.

(5.2) $p \equiv 19 \pmod{24}$. If $t \geq 2$, $[p,0,p]$ is an exceptional core $\mathbb{Z}$-lattice. So we may assume that $t = 1$. Assume that $5 \sim 1$. If $e \geq 520$, then $[520, 65, \frac{5(p+13)}{8}]$ is an exceptional core $\mathbb{Z}$-lattice. For the remaining cases, we can easily check the following table similar to (T).

$$\begin{array}{l} L[499] : [13,3,231],\quad L[379] : [13,1,175],\quad L[331] : [10,2,199] \\ L[211] : [15,3,85],\quad L[139] : [10,4,85],\quad L[19] : [13,4,10]. \end{array}$$

Note that any binary $\mathbb{Z}$-lattice ℓ such that $\ell_2 \simeq [5,0,10]$ and $\ell_3 \simeq [1,0,6]$ with $d\ell = 6p$ can be a core $\mathbb{Z}$-lattice of $L(e)$. This makes it easy to check the above table. If $5 \sim \Delta$ and $e \geq 330$, then $[330, 55, \frac{5(p+11)}{6}]$ is an exceptional core $\mathbb{Z}$-lattice. For the remaining cases, we have the followings:

$$\begin{array}{l} L[307] : [13,2,142],\quad L[283] : [37,2,46],\quad L[187] : [13,3,87] \\ L[163] : [7,3,141],\quad L[67] : [13,1,31],\quad L[43] : [7,1,37]. \end{array}$$

(5.3) $p \equiv 7 \pmod{24}$. Similarly to (5.2), we may assume that $t = 1$. If $e \geq 66$, $[66, 11, \frac{p+11}{6}]$ is an exceptional core $\mathbb{Z}$-lattice and $[10,2,19]$ is an exceptional core $\mathbb{Z}$-lattice of $L(31)$. $L(7)$ is a candidate.

(5.4) $t \sim \Delta$. Note that $t \geq 2$. If $p \equiv 7 \pmod{12}$, then $[p,0,p]$ is an exceptional core $\mathbb{Z}$-lattice and if $p \equiv 5 \pmod{12}$, $[2p,p,2p]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$.

Case 6 $L(e) := L' \perp \langle e\rangle = \langle 1,1,2,3,e\rangle$

Note that $[1,0,10] \simeq [3,0,14] \not\longrightarrow (L')_2$ and $[2,0,6] \not\longrightarrow (L')_3$. If $e \equiv 1 \pmod 3$, then $[2,1,2]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$ with a core prime 3. If $e \equiv 0 \pmod 3$ and $e \geq 6$, then $[6,3,3a+2] \not\longrightarrow L(e)$ for any non-negative integer a. Note that $[2,1,2]$ is an exceptional core $\mathbb{Z}$-lattice of $L(3)$. Now we always assume that $e \equiv 2 \pmod 3$.

Assume that 2 ia a core prime of $L(e)$, i.e., $e = 2^{2k}(8n+5),\ 2^{2k}(8n+7)$, $2^{2k+1}(8n+1)$ or $2^{2k+1}(8n+3)$, where $k = 0, 1$. We give the following table of an exceptional core $\mathbb{Z}$-lattice of each right $\mathbb{Z}$-lattice:

$$\begin{array}{lcllcl} e \equiv 5 \pmod 8,\ e \geq 13 & : & [14,0,12], & e \equiv 7 \pmod 8 & : & [10,0,4], \\ \frac{e}{2} \equiv 1 \pmod 8 & : & [43,2,4], & \frac{e}{2} \equiv 3 \pmod 8 & : & [9,2,12], \\ \frac{e}{4} \equiv 5 \pmod 8 & : & [14,6,14], & \frac{e}{4} \equiv 7 \pmod 8 & : & [10,0,16], \\ \frac{e}{8} \equiv 1 \pmod 8,\ e \geq 9 & : & [43,4,16], & \frac{e}{8} \equiv 3 \pmod 8 & : & [41,2,4]. \end{array}$$

$L(5)$ and $L(8)$ are not yet determined whether they are almost 2-universal or not.

If 2 is not a core prime of $L(e)$, then there exists a core prime $p \neq 2, 3$ dividing e. Clearly $p \equiv \pm 7, \pm 11 \pmod{24}$. We let $e = pt$.

(6.1) $p \equiv 7 \pmod{24}$. Since $e \equiv 2 \pmod 3$, $t \neq 1$. Therefore $[p,0,p]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$.

(6.2) $p \equiv 11 \pmod{24}$. In this case, $[2p,p,2p]$ is always an exceptional core $\mathbb{Z}$-lattice.

(6.3) $p \equiv 17 \pmod{24}$. Note that $2 \sim 1$ and $3 \sim \Delta$. If $t \sim \Delta$, $[10,0,p]$ is an exceptional core $\mathbb{Z}$-lattice. Assume that $t \sim 1$. If $t \geq 4$, $[14,0,3p]$ is an exceptional core $\mathbb{Z}$-lattice. Hence we may assume that $t = 1$. If $e \neq 17$, $[30,10,\frac{p+10}{3}]$ is an exceptional core $\mathbb{Z}$-lattice. Since $3, 5, 7$ are all nonsquare units in $\mathbb{Z}_{17}^*$, $[34,17,68]$ is an exceptional core $\mathbb{Z}$-lattice of $L(17)$.

(6.4) $p \equiv 13 \pmod{24}$. Note that $2 \sim \Delta$ and $3 \sim 1$. Assume that $t \sim \Delta$. If $t \geq 3$, then $[p,0,2p]$ is an exceptional core $\mathbb{Z}$-lattice. Hence we may assume that $t = 2$. If $7 \sim 1$ and $e \geq 308$, $[308,77,\frac{7(p+11)}{4}]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$. Note that $[11,2,71]$ is an exceptional core $\mathbb{Z}$-lattice of $L(74)$ and $[109,0,1090]$ is an exceptional core $\mathbb{Z}$-lattice of $L(218)$. If $7 \sim \Delta$ and $e \geq 210$, then $[210,35,\frac{7(p+5)}{6}]$ is an exceptional core $\mathbb{Z}$-lattice. Furthermore $[13,0,91]$ is an exceptional core $\mathbb{Z}$-lattice of $L(26)$ and $[61,0,610]$ is an exceptional core $\mathbb{Z}$-lattice of $L(122)$.

If $t \sim 1$, then $[6,3,\frac{p+3}{2}]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$.

Case 7 $L(e) := L' \perp \langle e \rangle = \langle 1,1,2,5,e \rangle$

Note that $[2,0,3] \simeq [5,0,14] \not\longrightarrow (L')_2$ and $[1,0,10] \not\longrightarrow (L')_5$. If $e \equiv \pm 1 \pmod 5$, which implies that 5 is a core prime of $L(e)$, $[4,1,4]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$. Note that $[4,1,4]$ is represented by $\langle 1,1,2,5 \rangle$ over all p-adic integers but it is not represented by $\langle 1,1,2,5 \rangle$. This is possible for the fact that the class number of $\langle 1,1,2,5 \rangle$ is 2 (see [17]). If $e \equiv 0 \pmod 5$ and $e \geq 15$, then $[10,0,5a+1] \not\longrightarrow L(e)$. $[2,1,4]$ is an exceptional core $\mathbb{Z}$-lattice of $L(10)$ and $[15,0,5a+1] \not\longrightarrow L(5)$. Now we always assume that $e \equiv \pm 2 \pmod 5$.

Assume that 2 is a core prime of $L(e)$, i.e., $e = 2^{2k}(8n \pm 3)$ or $2^{2k+1}(8n \pm 1)$, where $k = 0, 1$. Similarly to Case (6), we have the following table:

$$\begin{array}{llll} e \equiv 3 \pmod 8 : [14,0,20], & e \equiv 5 \pmod 8 : [2,0,12], \\ \frac{e}{2} \equiv 1 \pmod 8 : [5,1,5], & \frac{e}{2} \equiv 7 \pmod 8 : [3,0,8], \\ \frac{e}{4} \equiv 3 \pmod 8 : [14,4,4], & \frac{e}{4} \equiv 5 \pmod 8 : [2,0,48], \\ e \equiv 0 \pmod 8 : [2,0,8a+3] \not\longrightarrow L(e). \end{array}$$

We consider the remaining case. Let $p \neq 2, 5$ be a core prime of $L(e)$. Note that $p \equiv \pm 7, \pm 11, \pm 17, \pm 19 \pmod{40}$. Since p divides e, we let $e = pt$.

(7.1) $p \equiv 11, 19 \pmod{40}$. Since $t \neq 1$, $[p, 0, p]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$.

(7.2) $p \equiv 7 \pmod{40}$. If $t \sim \Delta$, $[2p, 0, 5]$ is an exceptional core $\mathbb{Z}$-lattice. Assume that $t \sim 1$. Then we may assume that $t = 1$ by a similar reasoning to (7.1). If $7 \sim 1$ and $e \geq 210$, then $[210, 21, \frac{7(p+3)}{10}]$ is an exceptional core $\mathbb{Z}$-lattice. Furthermore $[27, 4, 18]$ is an exceptional core $\mathbb{Z}$-lattice of $L(47)$ and $[44, 1, 19]$ is an exceptional core $\mathbb{Z}$-lattice of $L(167)$. If $7 \sim \Delta$, then $[56, 7, \frac{7(p+1)}{8}]$ is an exceptional core $\mathbb{Z}$-lattice of $L(e)$. Since $[14, 7, 14]$ is not represented by L, it is an exceptional core $\mathbb{Z}$-lattice of $L(7)$.

(7.3) $p \equiv 17 \pmod{40}$. Note that $2 \sim 1$ and $5 \sim \Delta$. If $t \sim \Delta$, then $[2p, 0, 3]$ is an exceptional core $\mathbb{Z}$-lattice. Assume that $t \sim 1$. If $e \geq 190$, then $[190, 38, \frac{p+38}{5}]$ is an exceptional core $\mathbb{Z}$-lattice. If $t \geq 6$, $[5p, 0, 14]$ is an exceptional core $\mathbb{Z}$-lattice. Therefore we may assume that $t = 1$ or 4. For the remaining cases, we have:

$$L(17), L(68) : [34, 17, 68], \quad L(97) : [9, 1, 54], \quad L(137) : [14, 1, 49].$$

(7.4) $p \equiv 21 \pmod{40}$. If $t \sim \Delta$, $[p, 0, 14]$ is an exceptional core $\mathbb{Z}$-lattice and if $t \sim 1$, then $t \geq 3$ and hence $[p, 0, 2p]$ is an exceptional core $\mathbb{Z}$-lattice.

(7.5) $p \equiv 23 \pmod{40}$. If $t \sim \Delta$, $[2p, 0, 5]$ is an exceptional core $\mathbb{Z}$-lattice. Assume that $t \sim 1$. If $t \geq 6$, $[5p, 0, 1]$ is an exceptional core $\mathbb{Z}$-lattice. So we may assume that $t = 1$ or 4. If $e \geq 210$, $[210, 42, \frac{p+42}{5}]$ is an exceptional core $\mathbb{Z}$-lattice. For the remaining cases, we have:

$$L(23), L(92) : [13, 2, 18], \quad L(143) : [18, 8, 83], \quad L(103) : [13, 6, 82].$$

(7.6) $p \equiv 29 \pmod{40}$. If $t \sim \Delta$, $[p, 0, 10]$ is an exceptional core $\mathbb{Z}$-lattice and if $t \sim 1$, $[p, 0, 2p]$ is an exceptional core $\mathbb{Z}$-lattice.

(7.7) $p \equiv 33 \pmod{40}$. If $t \sim \Delta$, $[2p, 0, 3]$ is an exceptional core $\mathbb{Z}$-lattice. Assume that $t \sim 1$. Then we may assume that $t = 1$ or 4 by a similar reasoning to the subcase (7.5). If $e \geq 110$, $[110, 22, \frac{p+22}{5}]$ is an exceptional core $\mathbb{Z}$-lattice. Lastly, $[13, 2, 34]$ is an exceptional core $\mathbb{Z}$-lattice of $L(73)$.

Theorem 3.1 *The following left quinary $\mathbb{Z}$-lattices represent all binary $\mathbb{Z}$-lattices except the following right ones:*

Quinary $\mathbb{Z}$-lattices	:	*Exceptional binary $\mathbb{Z}$-lattices*
$L := \langle 1, 1, 2, 2, 5 \rangle$	:	$[2, 1, 2]$, $[2, 1, 4]$, $[4, 1, 4]$, $[8, 1, 8]$,
$\langle 1, 1, 1, 1, 5 \rangle$	:	$[2, 1, 4]$, $[4, 1, 4]$, $[8, 1, 8]$.

Proof. Since the first quinary $\mathbb{Z}$-lattice is the sublattice of the second one, it suffices only to prove the first case. Note that $K := \langle 1,1,2,2\rangle$ has class number one. Let $\ell := [a,b,c]$ be a Minkowski reduced binary $\mathbb{Z}$-lattice. We assume that ℓ is 2-primitive $\mathbb{Z}$-lattice, i.e., every $\mathbb{Z}_2$-lattice on $\mathbb{Q}_2\ell$ containing ℓ_2 is isometric to ℓ_2. If $\ell_2 \not\simeq [2,1,2]$ and if $\mathbb{Q}_2\ell$ is not a hyperbolic space, then $\ell \longrightarrow K$. First assume that $\ell_2 \simeq [2,1,2]$. If $c \geq 7$, $[a,b,c-5] \longrightarrow K$. Therefore $[2,1,2]$ is the only one exception of this case. Now assume that $\mathbb{Q}_2\ell$ is a hyperbolic space. By a direct calculation, we can easily check that at least one of the following $\mathbb{Z}$-lattices

$$[a-5,b,c], \qquad [a,b,c-5], \qquad [a-5,b-5,c-5]$$

is neither isometric to $[2,1,2]$ over $\mathbb{Z}_2$ nor hyperbolic space over $\mathbb{Q}_2$. If $a \geq 14$, all of $\mathbb{Z}$-lattices given above are positive definite and hence $\ell \longrightarrow \langle 1,1,2,2,5\rangle$. Now we assume that $a \leq 13$. Since the other cases can be done in a similar manner, we consider only $a = 2,4,8$. Assume that $a = 2$. Since $\mathbb{Q}_2\ell$ is hyperbolic and ℓ_2 is primitive, $b = 1$ and $c \equiv 0 \pmod 4$. Therefore if $c \geq 8$, $[2,1,c-5] \longrightarrow K$ and hence $\ell \longrightarrow L$. Clearly $[2,1,4]$ is not represented by L. If $a = 4$, then $b = 1$ and $c \equiv 0 \pmod 2$ by a similar reasoning to above. If $c \geq 6$, $[a,b,c-5] \longrightarrow K$ and hence $\ell \longrightarrow L$. $[4,1,4]$ is an exception. If $a = 8$, then b must be 1 or 3. Hence if $c \geq 11$, at least one of the $[3,b,c]$, $[3,b-5,c-5]$ is represented by K, which implies $\ell \longrightarrow L$. For the remaining cases, we can easily check that L represents all except $[8,1,8]$ by a direct calculation. Assume that ℓ is not 2-primitive. It suffices to check the case when ℓ is a sublattice of one of the exceptional $\mathbb{Z}$-lattices with even index, which is given above. The all sublattices with index 2 of the $[2,1,2], [2,1,4], [4,1,4], [8,1,8]$ are

$$[2,0,6], \quad [4,2,8], \quad [2,0,14], \quad [4,2,16], \quad [6,0,10], \quad [8,2,32], \quad [14,0,18].$$

One can easily check that these are all represented by L by a direct calculation. Therefore the desired result follows.

Theorem 3.2 *The quinary $\mathbb{Z}$-lattice $L := \langle 1,1,1,2,7\rangle$ represents all binary $\mathbb{Z}$-lattices except* $[3,0,3], [6,0,6]$.

Proof. Note that the quaternary sublattice $K := \langle 1,1,1,2\rangle$ of L has class number 1. Let $\ell := [a,b,c]$ be a Minkowski reduced binary 2-primitive $\mathbb{Z}$-lattice. The idea of the proof is similar to that of [11]. So we provide only a sketch of the proof. Note that

$$\ell_p \longrightarrow K_p \ \text{ if } \begin{cases} p \equiv \pm 1 \pmod 8 \text{ or} \\ p \equiv \pm 3 \pmod 8 \text{ and } \gcd(p,a,b,c) = 1 \text{ or} \\ p = 2 \text{ and } \ell_2 \not\simeq [2,0,7] \simeq [1,0,14]. \end{cases}$$

In particular, if ℓ_2 is unimodular, then $\ell_2 \longrightarrow K_2$. We let

$$\ell_s(t) := [a-7t^2, sa+b, s^2a+2sb+c] = \begin{pmatrix} a-7t^2 & sa+b \\ sa+b & s^2a+2sb+c \end{pmatrix}.$$

If $\ell_s(t) \longrightarrow K$, then $\ell \longrightarrow L$. Note that $\det(\ell_s(t)) = ac - b^2 - 7t^2(s^2a + 2sb + c)$. If $3a - 28t^2(s^2 + \frac{|s|+s}{2} + 1) > 0$ and $a > 7t^2$, then we can easily check that $\ell_s(t)$ is positive definite from the fact that $[a, b, c]$ is Minkowski reduced. Let $\mathfrak{P} = \{3, 5, 11, 13, 19, 29, 37 \dots\}$ be the set of primes p such that $p \equiv \pm 3 \pmod 8$.

Case (1) $a \equiv 2, 4 \pmod 8$. For any integer s, $(\ell_s(1))_2 \longrightarrow K_2$. Let $p_1, p_2, \dots, p_k$ be the primes in $\mathfrak{P}$ dividing $a-7$. Note that $a-7 \geq p_1p_2\cdots p_k$. If $k = 0$ and $a \geq 10$, then $\ell_0(1) \longrightarrow K$. Assume that $a = 2$. If $b = 0$, the desired representation follows from the fact that $\langle 1, 1, 1, 7\rangle$ is 1-universal. If $b = 1$, $[2, 1, c] \longrightarrow K$. Since ℓ is 2-primitive, if $a = 4$, then $b = 1$. Clearly $[4, 1, c] \longrightarrow K$. If $k = 1$ and $a \geq 19$, then $\ell_0(1)$ or $\ell_{-1}(1)$ is represented by K. For $a = 10, 12, 18$, we can easily show that $\ell \longrightarrow L$ by a direct calculation. If $2 \leq k \leq 5$, for at least one integer s in $\{-k+1, \dots, -1, 0, 1, \dots, k-1\}$, $\ell_s(1) \longrightarrow K$. If $k = 6$, for at least one integer s in $\{-7, -6, \dots, 6, 7\}$, $\ell_s(1) \longrightarrow K$. If $k \geq 7$, for at least one integer s in $\{-k2^{k-1}, -k2^{k-1} + 1, \dots, 0, 1, \dots, k2^{k-1}\}$, $\ell_s(1) \longrightarrow K$ by Lemma 3 of [10].

Case (2) $a \equiv 0 \pmod 8$. For any integer s, $(\ell_s(2))_2 \longrightarrow K_2$. All other things are similar to above case. In this case, it suffices to check only the cases when $a = 8$, 16, 24, 32, 40, 48, 72, 88 by a direct calculation.

Case (3) $a \equiv 6 \pmod 8$. If $b \equiv 0 \pmod 2$, then $c \equiv 1 \pmod 2$ by a 2-primitiveness assumption of ℓ. Therefore $(\ell_s(1))_2 \longrightarrow K_2$. If $b \equiv 1 \pmod 2$, then $(\ell_s(2))_2 \longrightarrow K_2$. All other things are similar to Case (1).

Case (4) $a \equiv 1 \pmod 2$. If $b \equiv 0 \pmod 2$, $(\ell_s(1))_2 \longrightarrow K_2$ for all odd integers s and if $b \equiv 1 \pmod 2$, $(\ell_s(1))_2 \longrightarrow K_2$ for all even integers s. Since all other things are similar to Case (1), we consider only $a = 3$. Since $[3, b, c] \longrightarrow K_2$, it suffices to check only the case when $b = 0$ and $c \equiv 0 \pmod 3$. If $c \geq 8$, then $[3, 0, c-7] \longrightarrow K$ and $[3, 0, 6] \longrightarrow K$. Therefore $[3, 0, 3]$ is the only one exception.

Lastly, we must check the representation of the sublattices $[3, 0, 3]$ with even index by L. Let ℓ' be such a lattice. Since only $[3, 0, 12], [6, 0, 6]$ are all sublattices of $[3, 0, 3]$ with index 2 and $[3, 0, 12] \longrightarrow L$, we may assume that $\ell' \longrightarrow [6, 0, 6]$. If ℓ' has an even index in $[6, 0, 6]$, then $\ell' \longrightarrow [6, 0, 24]$ or $\ell' \longrightarrow [12, 0, 12]$. Therefore $\ell' \longrightarrow L$. Assume that $\ell' := [6a, 6b, 6c]$ be a sublattice of $[6, 0, 6]$ with an odd index. Then $ac - b^2$ must be an odd square integer. We define

$$\ell'_s(t) = [6a - 7t^2, 6as + 6b, 6as^2 + 12sb + 6c] = \begin{pmatrix} 6a - 7t^2 & 6as + 6b \\ 6as + 6b & 6as^2 + 12sb + 6c \end{pmatrix}.$$

If $a \equiv 0 \pmod 2$, then $a \equiv 2 \pmod 4$. Therefore $(\ell'_s(1))_2 \longrightarrow K_2$. If $a \equiv 1 \pmod 2$, $\mathrm{ord}_2(\det(\ell'_s(2))) = 2$. Therefore $(\ell'_s(2))_2 \longrightarrow K_2$. By a similar calculation to Case (1), we can easily check the desired representation for all cases.

To sum up all, we get the following theorem:

Theorem 3.3 *The all quinary diagonal almost* 2*-universal* $\mathbb{Z}$*-lattices and its exceptions are the followings:*

(1) 2*-universal* $\mathbb{Z}$*-lattices*

$$\langle 1,1,1,1,a\rangle \quad a=1,2,3, \qquad \langle 1,1,1,2,b\rangle \quad b=2,3.$$

(2) Almost 2*-universal* $\mathbb{Z}$*-lattices and its exceptions*

$$\begin{array}{lll} \langle 1,1,1,2,4\rangle : [3,0,3], & \langle 1,1,1,1,5\rangle : [2,1,4],\ [4,1,4],\ [8,1,8], \\ \langle 1,1,1,2,5\rangle : [3,0,3], & \langle 1,1,1,2,7\rangle : [3,0,3],\ [6,0,6], \\ \langle 1,1,2,2,3\rangle : [2,1,2], & \langle 1,1,2,2,5\rangle : [2,1,2],\ [2,1,4],\ [4,1,4],\ [8,1,8]. \end{array}$$

(3) Candidates

$$\langle 1,1,1,3,7\rangle,\quad \langle 1,1,2,3,5\rangle,\quad \langle 1,1,2,3,8\rangle.$$

Corollary 3.4 *For* $n \geq 3$*, The all diagonal almost* n*-universal* $\mathbb{Z}$*-lattices of rank* $n+3$ *are, in fact,* n*-universal except only* $\langle 1,1,1,1,1,2,2\rangle$*. They are:*

(1) $n=3$

$$\langle 1,1,1,1,1,a\rangle, \quad a=1,2,3, \quad \langle 1,1,1,1,2,b\rangle \quad b=2,3.$$

(2) $n=4$

$$\langle 1,1,1,1,1,1,1\rangle, \quad \langle 1,1,1,1,1,1,2\rangle, \quad \langle 1,1,1,1,1,2,2\rangle \ : \ \textit{Candidate}\ .$$

(3) $n=5$

$$\langle 1,1,1,1,1,1,1,1\rangle \quad \langle 1,1,1,1,1,1,1,2\rangle.$$

Proof. The universality of the above $\mathbb{Z}$-lattices are given by [10]. Let L be a diagonal almost 3-universal $\mathbb{Z}$-lattice of rank 6. Then $L \simeq \langle 1\rangle \perp L'$, where L' is the almost 2-universal quinary $\mathbb{Z}$-lattice. Among them, $\langle 1,1,1,1,2,4\rangle$ and $\langle 1,1,1,2,3,8\rangle$ are not locally 3-universal. Since $\langle 2\rangle^{\perp}$ in L is also almost 2-universal, L cannot be

$$\langle 1,1,1,1,2,7\rangle,\ \ \langle 1,1,1,1,3,7\rangle \quad \text{and} \quad \langle 1,1,1,2,3,5\rangle.$$

Furthermore one can easily check that

$$\begin{pmatrix} 5 & 2 & 2^m \\ 2 & 5 & -2^m \\ 2^m & -2^m & 2^{2m} \end{pmatrix} \not\longrightarrow \langle 1,1,1,1,1,5\rangle,$$

for all non-negative integers m and hence $\langle 1,1,1,2,2,5\rangle$ is not almost 3-universal. Finally, we can easily check that

$$[4,1,4]\perp\langle 2\cdot 5^{2m}\rangle \not\longrightarrow \langle 1,1,1,1,2,5\rangle, \quad [2,1,2]\perp\langle 3^{2m}\rangle \not\longrightarrow \langle 1,1,1,2,2,3\rangle.$$

Since there are 5 diagonal almost 3-universal $\mathbb{Z}$-lattices of rank 6, we have 5 candidates of diagonal almost 4-universal $\mathbb{Z}$-lattices of rank 7. Since $[2,1,2]^{\perp}$ in $\langle 1,1,1,1,1,1,3\rangle$ or in $\langle 1,1,1,1,1,2,3\rangle$ is not almost 2-universal, both $\mathbb{Z}$-lattices are not almost 4-universal. $\langle 1,1,1,1,1,2,2\rangle$ is a candidate. Note that $[2,1,2]\perp[2,1,2]$ is not represented by $\langle 1,1,1,1,1,2,2\rangle$ and

$$[2,1,2]\perp[2,1,2]\perp\langle 3^{2m}\rangle \not\longrightarrow \langle 1,1,1,1,1,1,2,2\rangle.$$

Remark 3.5 *Let L be an almost 6-universal $\mathbb{Z}$-lattice and let I_k be the $\mathbb{Z}$-lattice generated by the vectors of quadratic norm 1 of rank k. Since $I_5 \longrightarrow L$, $L \simeq I_n \perp L'$, where $\langle 1\rangle \not\longrightarrow L'$ and $n \geq 5$. For $n \geq 2$, since $D_5(16n-20)[1\frac{1}{4}]$ (for the notation, see [4]) is not represented by I_k for all k, at least one of the $\mathbb{Z}$-lattices of the form $D_5(16n-20)[1\frac{1}{4}]$ must be represented by L'. Under this restriction, if the rank of L is 11, then for all $n \geq 1$, $A_5(36n-30)[1\frac{1}{6}]$ is not represented by L. Therefore the rank of L is greater than 11.*

If we define $au_{\mathbb{Z}}(n)$ to be the minimal rank of an almost n-universal $\mathbb{Z}$-lattice, then $au_{\mathbb{Z}}(n) \geq u_{\mathbb{Z}}(n-1)$ (for the definition, see [10]). Therefore $au_{\mathbb{Z}}(n)$ grows very quickly.

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