Contemporary Mathematics
Volume **00**, XXXX

Regularity Properties of Positive Definite Integral Quadratic Forms

Wai Kiu Chan, A. G. Earnest and Byeong-Kweon Oh

Abstract. This paper presents an overview of recent results on positive definite integral quadratic forms which satisfy various regularity properties, and extends the results to obtain new finiteness theorems for regular quadratic forms which do not contain proper regular subforms.

Introduction

The goal of this paper is to describe and extend results on integral quadratic forms satisfying various regularity properties, emphasizing the common threads unifying some recent papers on this subject, as presented in the authors' three talks at the Talca conference.

In terminology first introduced by Dickson in 1927, an integral quadratic form is said to be *regular* if it globally represents all integers that are locally represented everywhere by the form. It has been known since early work of Watson that the number of equivalence classes of regular positive definite primitive integral ternary quadratic forms is finite (for convenience, forms with all these properties will be referred to simply as regular ternary forms). In his unpublished thesis [**15**], Watson established this finiteness by utilizing certain regularity-preserving transformations, in conjunction with numerous explicit calculations, to produce bounds for the possible prime power divisors of the discriminants of regular ternary forms. Watson subsequently published a more far-reaching result showing that the size of the exceptional set of integers that are represented everywhere locally, but not globally, by a positive definite primitive integral ternary quadratic form grows asymptotically as a function of the discriminant of the form. While establishing the finiteness of the number of regular ternary forms (indeed for the ternary forms having exceptional set of any prescribed size), this result relies for its proof on various character sum estimates and is consequently not computationally effective.

1991 *Mathematics Subject Classification.* Primary 11E12.

Key words and phrases. Integral quadratic forms.

The work of the first author was supported in part by NSF grant #DMS-0138524 and NSA grant #MDA904-02-1-0055.

The work of the second author was supported in part by NSF grant #DMS-9988672.

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More recently, Kaplansky revived interest in completing the problem of finding representatives of all classes of regular ternary forms. This work culminated in the paper [**8**], which presents a list containing representatives from all such classes. Of the 913 forms in the list, regularity is proved for all but 22 forms, which remain only "candidates". For the results and proofs stated later in this paper, properties of the forms in this list will be freely utilized. In particular, by an examination of the forms in this list, or by referring to results in Watson's thesis, it can be seen that all prime divisors of the discriminants of regular ternary forms lie in the set $\mathcal{S} = \{2, 3, 5, 7, 11, 13, 17, 23\}$, that only certain combinations of these primes can occur in the same discriminant, and that no prime exceeding 3 can occur in the discriminant to a power exceeding 3.

In order to produce the list of regular ternary forms in [**8**], the authors of that paper returned to the regularity-preserving transformations used by Watson in his thesis. The main theme of the present paper is to show how these transformations can be further exploited to obtain computationally effective results for forms satisfying various generalizations of the original regularity condition, as well as to establish finiteness results in a broader context. The approach to be taken here will be to carefully state representative results, but to give only outlines of proofs, or to describe special cases that illustrate the salient features of the general arguments, and refer the reader to the original papers [**2**], [**3**], [**4**] and [**11**] for further details and more extensive references to prior literature. Moreover, the methods of those papers will be extended to prove new finiteness results for regular forms which contain no proper regular subforms.

1. Preliminaries

Throughout this paper, the geometric language of quadratic spaces and lattices will be adopted, following [**13**]. So the objects of study will be R-lattices on a nondegenerate quadratic space (V, Q) over a field F, where R is the ring of integers $\mathbb{Z}$ or one of its local completions $\mathbb{Z}_p$, where p is a prime, and F is the quotient field of R. When $R = \mathbb{Z}$, it will be assumed throughout the paper that the underlying rational quadratic space (V, Q) is positive definite. In order to obtain finiteness results for the number of inequivalent $\mathbb{Z}$-lattices with exceptional set of a specified size, it is necessary to further impose some primitivity condition on the lattices under consideration. Throughout the remainder of the paper, it will be assumed that the $\mathbb{Z}$-lattices under consideration satisfy the primitivity condition $\mathfrak{n}L = 2\mathbb{Z}$, where $\mathfrak{n}L$ denotes the norm ideal of L. Consequently, the scale ideal $\mathfrak{s}L$ of L is either $\mathbb{Z}$ or $2\mathbb{Z}$. For brevity, *the term "$\mathbb{Z}$-lattice" will be reserved from this point on for a lattice satisfying all of the conditions of this paragraph.*

To see the connection with the terminology for quadratic forms used in more classical literature, let $f = \sum_{1 \le i \le j \le \ell} a_{ij} x_i x_j$, with $a_{ij} \in \mathbb{Z}$, be an integral quadratic form. Associate to f the matrix $M_f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) = (a'_{ij})$ (that is, $a'_{ii} = 2a_{ii}$ and $a'_{ij} = a'_{ji} = a_{ij}$ for $i < j$). Let V_f be the rational vector space spanned by vectors $e_1, \dots, e_\ell$ equipped with the symmetric bilinear form B_f for which $B_f(e_i, e_j) = a'_{ij}$, and the corresponding quadratic map $Q_f(v) = B_f(v, v)$. Then $L_f = \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_\ell$ is a $\mathbb{Z}$-lattice on V_f for which $\mathfrak{n}L_f \subseteq 2\mathbb{Z}$. The original form f is primitive in the sense that $\text{g.c.d.}_{1 \le i \le j \le \ell}\{a_{ij}\} = 1$ if and only if $\mathfrak{n}L_f = 2\mathbb{Z}$. Such a primitive form f is classic integral if and only if $\mathfrak{s}L_f = 2\mathbb{Z}$; otherwise, $\mathfrak{s}L_f = \mathbb{Z}$ and f is non-classic integral (sometimes referred to as integer-valued). An integer a is represented by the form f if and only if $2a$ is represented by the lattice L_f. Finally, if f is a ternary form and df is the discriminant of f used in [**15**], then $dL_f = 2df$.

For a $\mathbb{Z}$-lattice L and a positive integer n, let $Q_n(L)$ denote the set of all $\mathbb{Z}$-lattices of rank n that are represented by L, and let $Q_n(\operatorname{gen} L)$ and $Q_n(\operatorname{spn} L)$ denote the corresponding sets of lattices of rank n that are represented by the genus or spinor genus of L, respectively. For simplicity, the subscript will be omitted when $n = 1$. Note that an integer $2a$ lies in $Q(\operatorname{gen} L)$ if and only if a is an *eligible integer* for the corresponding quadratic form in the terminology of Kaplansky.

This paper will be concerned with the exceptional sets $E_n(L) = Q_n(\operatorname{gen} L) \setminus Q_n(L)$ and the spinor exceptional sets $E_n^s(L) = Q_n(\operatorname{spn} L) \setminus Q_n(L)$. Subscripts will also be omitted here when $n = 1$. In this notation, an integral quadratic form is regular in the sense of Dickson precisely when $E(L)$ is empty for the corresponding $\mathbb{Z}$-lattice L. For an arbitrary positive integer n, a lattice L will be said to be *n-regular* when $E_n(L)$ is empty, and *spinor n-regular* when $E_n^s(L)$ is empty.

The $\mathbb{Z}$-lattice L will be said to be *almost regular* when $E(L)$ is finite. It is well known that all $\mathbb{Z}$-lattices of rank at least 5 are almost regular, and that those of rank 4 are as well if one restricts, for example, to the consideration of primitive representations. However, for ternary $\mathbb{Z}$-lattices L, the exceptional set $E(L)$ can be, and often is, infinite. The results of [**16**] assert that the size of the set $E(L)$ grows asymptotically as a function of the discriminant dL for primitive ternary $\mathbb{Z}$-lattices L. The corresponding result was established for the set $E^s(L)$ in [**1**]. However, the analytic methods of proof for these results rely on certain character sum estimates and are consequently not computationally effective for bounding the discriminants of those lattices L for which $E(L)$ or $E^s(L)$ is bounded by a prescribed constant.

The majority of this paper will focus on the case $n = 1$; that is, on the representation of integers. Following two sections containing background results, it will be shown in sections 4 and 5 how the Watson transformations can be utilized to produce computationally effective results for almost regular and spinor regular ternary $\mathbb{Z}$-lattices, respectively. Section 6 contains results on the determination of 2-regular quaternary lattices. Extensions of the finiteness results for regularity to the general context of the representation of one lattice by another are described in section 7. The final section of the paper, which is again restricted to the $n = 1$ case, contains a finiteness theorem for regular lattices of arbitrary rank which contain no proper sublattice for which the represented value set is sufficiently large.

2. Watson's transformations

The purpose of this section is to describe the transformations of Watson in the current geometric setting. Detailed proofs of the properties mentioned in this section can be found in [**2**], [**3**] or [**4**]. The description of the transformations, which will be denoted by λ_p, is based upon certain sublattices defined as follows:

Definition 2.1. For a lattice L (over either $\mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p) and a positive integer n, define

$$\Lambda_n(L) = \{x \in L : Q(x + z) \equiv Q(z) (\operatorname{mod} n), \text{ for all } z \in L\}.$$

Remark 2.2. The defining condition can equivalently be expressed as $Q(x) + 2B(x, z) \equiv 0 (\operatorname{mod} n)$. From this characterization, it is easily seen that $nv \in \Lambda_n(L)$, for all $v \in L$, and $Q(x) \equiv 0 (\operatorname{mod} n)$, for all $x \in \Lambda_n(L)$.

REMARK 2.3. If L is a $\mathbb{Z}$-lattice, then $\Lambda_{2p}(L)_p = \Lambda_{2p}(L_p)$ and $\Lambda_{2p}(L)_q = L_q$ for any prime $q \neq p$.

For several later results, it is useful to note the following elementary fact.

LEMMA 2.4. *If M is an anisotropic even unimodular $\mathbb{Z}_p$-lattice, then M cannot primitively represent any element of $2p\mathbb{Z}_p$.*

PROOF. Suppose there exists a primitive vector $x \in M$ for which $Q(x) \in 2p\mathbb{Z}_p$. By 82:17 of [**13**], there exists $y \in M$ such that $B(x, y) = 1$. But then $\mathbb{Z}_p x + \mathbb{Z}_p y$ is a hyperbolic plane contained in M, contrary to the assumption that M is anisotropic. $\square$

For the remainder of this section, L will denote a $\mathbb{Z}$-lattice and p a prime.

EXAMPLE 2.5. If $L_p = M \perp N$, where M is even unimodular and $sN \subseteq 2p\mathbb{Z}_p$, then $\Lambda_{2p}(L)_p = pM \perp N$.

EXAMPLE 2.6. If $sL_2 = 2\mathbb{Z}_2$, then $\Lambda_4(L_2) = \{x \in L_2 : Q(x) \equiv 0 (\text{mod } 4)\}$.

From the properties stated above, it can be verified that $\mathfrak{n}(\Lambda_{2p}(L))$ is either $2p\mathbb{Z}$ or $2p^2\mathbb{Z}$. Consequently, upon scaling the lattice $\Lambda_{2p}(L)$ by either p^{-1} or p^{-2}, a lattice whose norm ideal equals $2\mathbb{Z}$ is obtained. This resulting lattice will be denoted by $\lambda_p(L)$.

PROPOSITION 2.7. *Suppose that L_p has a Jordan splitting $M \perp N$, where M is the leading Jordan component of L_p and $\mathfrak{n}N \subseteq 2p^2\mathbb{Z}_p$.*

a) If M is anisotropic, then $|E(\lambda_p(L))| \leq |E(L)|$ and $|E^s(\lambda_p(L))| \leq |E^s(L)|$.

b) If M is isotropic, then $|E(\lambda_p(L))| \leq |E(L)|$.

PROOF. The result is easily established for the case when $p = 2$ and M is $2\mathbb{Z}_2$-modular, using Example 2.6. In all other cases, M is even unimodular.

a) Let $a \in Q(\text{gen}\, \Lambda_{2p}(L)) \subseteq Q(\text{gen}\, L)$. If $a \notin E(L)$, then there exists $v \in L$ such that $Q(v) = a$. Write $v = v_0 + v_1$, where $v_0 \in M$, $v_1 \in N$. Since $2p$ divides a by Remark 2.2, and $Q(v_1) \in 2p\mathbb{Z}_p$, it follows that $Q(v_0) \in 2p\mathbb{Z}_p$. So, by Lemma 2.4, $v_0 \in pM$, and $v \in \Lambda_{2p}(L)$ by Example 2.5. Thus, $a \notin E(\Lambda_{2p}(L))$. This establishes that $E(\Lambda_{2p}(L)) \subseteq E(L)$. As $|E(\Lambda_{2p}(L))| = |E(\lambda_p(L)|$ when these sets are finite, the claimed result follows. The same argument prevails with "genus" replaced by "spinor genus" throughout.

b) In this case, M is split by a hyperbolic plane $\mathbb{H}$. By Example 2.5, $\lambda_p(L)_p = (\Lambda_{2p}(L)_p)^{\frac{1}{p^2}} \cong M \perp N^{\frac{1}{p^2}}$; in particular, $\lambda_p(L)_p$ is split by $\mathbb{H}$. So $Q(\lambda_p(L)_p) = 2\mathbb{Z}_p = Q(L_p)$, and $\lambda_p(L)_q \cong L_q$ for all $q \neq p$; thus, $Q(\text{gen}\, \lambda_p(L)) = Q(\text{gen}\, L)$. Now suppose that $a \in Q(\text{gen}\, \lambda_p(L))$ but $a \notin E(L)$. Then there exists $v \in L$ such that $Q(v) = a$. Then $pv \in \Lambda_{2p}(L)$ and $p^2 a \in Q(\Lambda_{2p}(L))$. It follows that $a \in Q(\lambda_p(L))$. Thus, $E(\lambda_p(L)) \subseteq E(L)$.

$\square$

In particular, the above result shows that λ_p preserves regularity and almost regularity. An analogous result regarding n-regularity for $n \geq 2$ appears in Lemma 2.1 of [**3**].

3. Successive minima

For the remainder of the paper, the lattices under consideration will all be $\mathbb{Z}$-lattices. For such a lattice of rank ℓ, the successive minima $\mu_1(L), \ldots, \mu_\ell(L)$ can be defined as in Definition 2.1 of [**6**]. It is a fundamental fact that the product of these values satisfies the inequalities

$$dL \leq \mu_1(L)\cdots\mu_\ell(L) \leq CdL,$$

where $C = C(\ell)$ is a constant depending only on ℓ (for example, see Proposition 2.3 of [**6**]). Consequently, in order to obtain an upper bound for the discriminant dL it suffices to establish upper bounds for the successive minima of L. The following results will be crucial for that purpose.

LEMMA 3.1. *There exists a bound B such that for all regular lattices L of rank at least 4 and for all primes p exceeding B, $Q(L_p) = \mathbb{Z}_p$.*

PROOF. It follows from Corollary 3.4 of [**6**] that, for any choice of $\epsilon \in \{\pm 1\}$ there exists a bound B such that for all primes p exceeding B, there exist two relatively prime positive integers a, b (depending on p) such that $4ab < p$ and $(\frac{2a}{p}) = (\frac{2b}{p}) = \epsilon$.

Now let L be a regular lattice of rank at least 4, and let p be a prime exceeding B. By applying suitable λ_q-transformations, it may be assumed that $Q(L_q) = 2\mathbb{Z}_q$ holds for all $q \neq p$. There exists at least one choice of $\epsilon \in \{\pm 1\}$ such that L_p represents all integers c for which $(\frac{c}{p}) = \epsilon$. Since $p > B$, there exist relatively prime integers a, b such that $4ab < p$ and $2a, 2b \in Q(L_p)$. So $2a, 2b \in Q(\text{gen } L)$. Since L is regular, it follows that $2a, 2b \in Q(L)$. Thus, L contains a binary sublattice N such that $dN \leq 4ab < p$. Consequently, L_p has a unimodular Jordan component of rank at least 2, and it follows that $\mathbb{Z}_p^\times \subseteq Q(L_p)$, where $\mathbb{Z}_p^\times$ denotes the units of $\mathbb{Z}_p$. In particular, $2, 4 \in Q(\text{gen } L) = Q(L)$. Thus, $\mu_1(L) = 2$ and $\mu_2(L) \leq 4$. So L contains a binary sublattice isometric to one of [2,1,2], [2,0,2], [2,1,4] or [2,0,4] (where $[a, b, c]$ denotes the lattice with associated matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$). Each of these binary lattices fails to represent some integer not exceeding 10. This establishes a bound for $\mu_3(L)$ from which it can be shown that for sufficiently large primes p, the localization L_p is split by a unimodular sublattice of rank at least 3. This establishes the lemma. $\square$

COROLLARY 3.2. *There exists a constant C_3 such that $\mu_3(L) < C_3$ for all regular lattices L of rank at least 4.*

4. Almost regular ternaries

In this section and the next, L will always denote a ternary $\mathbb{Z}$-lattice. So the discriminant of L is an even positive integer. If p is any prime for which $2p^2$ divides dL, it can be shown that the power of p dividing $d\lambda_p(L)$ is strictly smaller than the power of p dividing dL (see Lemma 2.5 of [**2**]). So after repeatedly applying λ_p a finite number of times, one obtains a lattice whose discriminant is no longer divisible by $2p^2$, but for which the other prime power divisors of the discriminant are unchanged. This lattice will be denoted by $\lambda_{(p)}(L)$. Note that $\lambda_{(p)}(L)_p$ must be split by an even unimodular sublattice; thus, $Q(\lambda_{(p)}(L)_p) \supseteq 2\mathbb{Z}_p^\times$.

For a ternary $\mathbb{Z}$-lattice L, let $\mathcal{A}_L$ denote the set of primes such that the local space $\mathbb{Q}L_p$ is anisotropic. A computation of Hasse symbols shows that $\mathbb{Q}L_p$ is isotropic if and only if $S_p(\mathbb{Q}L) = (-1,-1)_p$. It then easily follows from the Hilbert Reciprocity Law that the set $\mathcal{A}_L$ contains an odd number of primes. If $p \in \mathcal{A}_L$, then the ambient space for the lattice $\lambda_{(p)}(L)$ is also anisotropic at p, since it is merely a scale of the original space. Consequently, if there existed an $a \in E(\lambda_{(p)}(L))$, then it would follow, using Lemma 2.4, that $p^{2m}a \in E(\lambda_{(p)}(L))$ for all $m \geq 0$. However, by repeated application of Proposition 2.7, $\lambda_{(p)}(L)$ is almost regular whenever L is almost regular. Consequently, if L is almost regular and $p \in \mathcal{A}_L$, then $\lambda_{(p)}(L)$ must be regular.

Now suppose that L is almost regular and let p be a prime divisor of dL. Suppose first that $p \notin \mathcal{A}_L$. Then there exists a prime $q \neq p$ in $\mathcal{A}_L$, and $\lambda_{(q)}(L)$ is a regular ternary lattice whose discriminant is divisible by the same power of p as dL. It follows that p lies in the set $\mathcal{S}$ of primes that occur as divisors of the discriminants of regular ternary lattices, and that the power of p occurring in dL is effectively bounded. On the other hand, suppose that $p \in \mathcal{A}_L$. Then, since $\lambda_{(p)}(L)$ is anisotropic at p, p must divide $d\lambda_{(p)}(L)$. Since $\lambda_{(p)}(L)$ is also regular, it again follows that $p \in \mathcal{S}$. In summary, if L is almost regular, then all prime divisors of dL lie in $\mathcal{S}$, and the power of any prime occurring in dL is effectively bounded, except possibly for the primes lying in the set $\mathcal{A}_L$.

In order to derive a characterization of almost regular ternary lattices, the spinor genus representation theory must be taken into consideration. Since L is ternary, it is known that an integer represented by the genus of L is represented either by all spinor genera in $\operatorname{gen} L$, or by exactly half. The latter occurs for the integers in at most finitely many squareclasses. A number $\hat{t}$ lying in such a squareclass is said to be a *primitive element* if all representations of $\hat{t}$ by the lattices in $\operatorname{gen} L$ are primitive. Let P_L denote the set of such primitive elements for the genus of L. See [**14**] for a proof that P_L is a finite set.

A characterization of almost regular ternaries can now be stated in the following proposition. The proof of this result appears in [**4**].

PROPOSITION 4.1. *Let L be a ternary lattice that is not regular. Then L is almost regular if and only if (i) $\mathcal{A}_L = \{p\}$ for some $p \in \{2, 3, 5, 7, 11, 13, 17\}$;(ii) $\lambda_{(p)}(L)$ is regular; and (iii) L represents all integers in P_L.*

REMARK 4.2. The primes in the list given in condition (i) above are simply the primes in $\mathcal{S}$ with 23 omitted because every regular ternary lattice is isotropic at 23, as can be seen from the lists appearing in [**8**] or [**2**].

This characterization leads to an effective method for determining the ternary lattices having exceptional set of any prescribed size.

THEOREM 4.3. *Given any nonnegative integer k, there is an effective upper bound (depending upon k) for the discriminants of the ternary lattices L for which $|E(L)| \leq k$.*

SKETCH OF PROOF. Let L be a ternary lattice for which $|E(L)| \leq k$. It can be assumed that L is not regular, so, by Proposition 4.1, $\mathcal{A}_L = \{p\}$ and $\lambda_{(p)}(L)$ is regular. By previous observations, it suffices to bound $\operatorname{ord}_p(dL)$. For this purpose, it can be assumed that $Q(L_q) = 2\mathbb{Z}_q$, for all $q \neq p$, by Proposition 2.7. Under these conditions, bounds for the successive minima of L can be established in a straightforward manner, resulting in the desired bound for the power of p dividing the discriminant. $\square$

REMARK 4.4. The effectiveness of the method sketched above is illustrated in [**4**] by determining explicit bounds for the case $k = 1$. The results are stated in Theorems 5.2 and 5.3 of that paper.

From the preceding results, it is clear that in any infinite family of almost regular ternary lattices, the size of the exceptional set must grow without bound. The following exhibits an infinite family of this type.

EXAMPLE 4.5. For a nonnegative integer n, let $K(n)$ be the lattice corresponding to the quadratic form $x^2+y^2+3^{2n+1}z^2$. All lattices in the family $\{K(n), n \geq 0\}$ are almost regular (see [**3**] or [**4**]).

5. Spinor regular ternaries

When the rank of a lattice L exceeds 3, it is well-known that every integer represented by $\operatorname{gen} L$ is represented by every spinor genus within that genus; consequently, every spinor regular lattice of rank exceeding 3 is regular. However, as noted in the previous section, for ternary lattices it can be the case that only half of the spinor genera in the genus represent a given integer represented by $\operatorname{gen} L$. Consequently, there can exist ternary lattices that are spinor regular (that is, they represent all values represented by their spinor genus) but not regular. All spinor regular ternary lattices that lie in genera containing more than one spinor genus and having discriminant not exceeding 2000 are enumerated in [**1**]. One example that arises there is of particular interest.

EXAMPLE 5.1. The lattice corresponding to the ternary form $3x^2+7y^2+7z^2-2yz+3xz+3xy$ is a spinor regular lattice of discriminant 864 which is not regular. It lies in a genus consisting of four classes, of which two lie in each of two spinor genera. One of the other forms in this genus is regular (and hence spinor regular), while the other two fail to be spinor regular.

The analytic methods used by Watson in [**16**] were extended in [**1**] to prove the asymptotic growth (with discriminant) of the size of the spinor exceptional set $E^s(L)$, for all primitive integral ternary lattices. In [**2**], Chan and Earnest utilize the λ_p-transformations to derive *a priori* bounds on the prime power divisors of the discriminants of spinor regular lattices, and to describe a method of determining all such lattices. The main results of that paper are summarized in the following theorem:

THEOREM 5.2. *Let L be a spinor regular ternary lattice. Then:*
a) there exist spinor regular lattices $L_1, \dots, L_t = L$ such that $\frac{1}{2}dL$ is squarefree and, for each $i = 1, \dots, t-1$, there exists a prime p_i and $k_i \in \{1,2,4\}$ such that $dL_{i+1} = p_i^{k_i} dL_i$; and
b) the prime divisors of dL lie in $\mathcal{S}$, and for each $p \in \mathcal{S}$ there exists an explicitly determined b_p such that $ord_p dL \leq b_p$.

OUTLINE OF PROOF. Starting from an arbitrary spinor regular L, successively apply λ_q-transformations for the prime divisors q of dL, repeating each until either the discriminant of the resulting lattice is no longer divisible by $2q^2$ or the resulting lattice has q-adic localization split by $\mathbb{H}$, whichever occurs first. Denote the resulting lattice by $\tilde{L}$. By Proposition 2.7, $\tilde{L}$ is still spinor regular (as are all the intermediate lattices). Moreover, the local spinor norm groups $\theta(O^+(\tilde{L}_q))$ all contain $\mathbb{Z}_q^\times$, so $\operatorname{spn} L = \operatorname{gen} L$. Thus, $\tilde{L}$ is regular. An analysis of possible local structures shows that the only case in which there is an odd prime divisor p of dL that no longer divides $d\tilde{L}$ is when the p-adic completion of the lattice K occurring after the next-to-final application of λ_p has a diagonal splitting of the type $\langle a, p^\beta b, p^\gamma c\rangle$ with

$a, b, c \in \mathbb{Z}_p^\times$, $\gamma = 2$ and either $\beta = 0$ or $\beta = 2$. However, in this case K_p contains a Jordan component of rank at least 2, and so $\theta(O^+(K_p)) \supseteq \mathbb{Z}_p^\times$. Let L' be the lattice with localizations $L'_p = K_p$ and $L'_q = \tilde{L}_q$ for all $q \notin p$. Then L' is regular, p divides dL', and p lies in $\mathcal{S}$. This establishes that $p \in \mathcal{S}$ for all prime divisors of dL.

To complete the proof, it remains to establish bounds for the prime powers dividing dL. For this purpose, it can be assumed that L is spinor regular, $p|dL$, and $Q(L_q) = 2\mathbb{Z}_q$ for all $q \neq p$. In particular, if q is any prime such that q does not divide dL and for which $2q \in Q(L_p)$, then $2q \in \operatorname{gen} L$. With the possible exception of the integers lying in one rational squareclass, these integers are represented by $\operatorname{spn} L$, and hence by L itself. If $2q_1$ and $2q_2$ are two such integers, then $4q_1q_2$ is a bound for the discriminant of the binary sublattice N of L spanned by vectors v_1, v_2 for which $\mu_i(L) = Q(v_i)$ for $i = 1, 2$. For each of the finitely many possibilities for N, an integer $2q$ can be found that is represented by L but not by N. In this way, a bound for dL is established. The power of p occurring in the discriminant of any spinor regular ternary lattice cannot exceed this bound.

□

REMARK 5.3. In Propositions 5.4 and 5.5 of [**2**], specific values of the bounds b_p are obtained for all primes in $\mathcal{S}$. In particular, for $p > 7$ it is shown that the power of p dividing the discriminant of a spinor regular ternary lattice never exceeds 3.

6. 2-regular quaternaries

The study of n-regular lattices for $n \geq 2$ was initiated by Earnest in [**6**], where it was proved that there exist only finitely many equivalence classes of 2-regular primitive integral quaternary lattices. The proof there uses an extension of the character sum techniques of Watson. By utilizing the λ_p-transformations, Oh [**11**] has recently completed the determination of all the 2-regular quaternary lattices with the property that $\mathfrak{s}L = \mathbb{Z}$. Throughout this section, all lattices L under consideration will be assumed to satisfy this condition on $\mathfrak{s}L$; such lattices will be referred to as "primitive even".

The first step in this determination is to find all primitive even 2-regular quaternary lattices that are *stable*, in a sense that will now be defined. For a prime p, the lattice L is said to be *p-stable* if either $\mathbb{H}$ splits L_p or $L_p \cong \mathbb{A} \perp \mathbb{A}^p$, where $\mathbb{A}$ denotes the binary anisotropic $\mathbb{Z}_p$-lattice $\langle 1, -\Delta_p \rangle$ when p is odd, and the lattice $A(2,2)$ when $p = 2$. The lattice L is called *stable* if L is p-stable for all primes p.

Now assume that L is 2-regular. If L is not p-stable, then $\lambda_p(L)$ is also 2-regular (see [**3**]). By successively applying the transformation λ_p a finite number of times, L can be transformed to a p-stable 2-regular $\mathbb{Z}$-lattice. The p-stable 2-regular $\mathbb{Z}$-lattice obtained from L in this way by applying the transformation λ_p a minimum number of times is denoted by $\lambda_{(p)}(L)$. The stable 2-regular $\mathbb{Z}$-lattice obtained from L by repeating the above process for all primes p is denoted by $\lambda(L)$ (see [**3**]). From the definition, $d(\lambda(L)) \mid dL$. Conversely, it can be shown that every prime divisor of $d(\lambda(L))$ which exceeds 3 is also a divisor of dL.

Let L be a stable $\mathbb{Z}$-lattice. Then, L represents all even integers. For a positive integer $k \leq 4$, let $L(k)$ denote the sublattice of L generated by the first k vectors in a Minkowski basis for L. Then, it is easily checked that $L(1) \cong \langle 2\rangle$ and

$$L(2) \cong [2,1,2],[2,0,2],[2,1,4] \quad \text{or} \quad [2,0,4].$$

Since $[2,1,2],[2,0,2],[2,1,4]$ and $[2,0,4]$ do not represent $4,6,6$, and 10, respectively, the third minimum $\mu_3(L)$ is not greater than this number in each case. Using the fact that L is 2-regular, it can then be proven that

$$\{p : p \mid dL\} \subset \mathcal{S} = \{2,3,5,7,11,13,17,23\},$$

and that there are exactly 48 distinct equivalence classes of stable 2-regular quaternary $\mathbb{Z}$-lattices.

Now, consider those 2-regular lattices L which fail to be p-stable for one prime p, but which are q-stable for all $q \neq p$. Then $p \in \mathcal{S}$ and, for each of these primes p, all $\mathbb{Z}$-lattices satisfying the above property can be enumerated, and all are found to have class number 1. As a sample, consider the case $p = 7$. Then L is isometric to $\lambda_7(K)$, where K is isometric to one of

$$[2,2,2,4,1,0,0,0,0,1], \quad [2,2,2,6,1,1,0,0,1,0], \quad [2,2,4,4,1,1,0,0,1,1],$$
$$[2,4,4,6,1,0,2,0,1,0] \quad \text{or} \quad [2,4,6,6,0,1,1,1,2,1],$$

or $L \cong [6,6,10,10,1,2,-2,2,-2,3]$. Here, $[a_1,a_2,a_3,a_4,b,c,d,e,f,g]$ represents the matrix

$$\begin{pmatrix} a_1 & b & c & e \\ b & a_2 & d & f \\ c & d & a_3 & g \\ e & f & g & a_4 \end{pmatrix}.$$

Finally, for an arbitrary $\mathbb{Z}$-lattice L, define $t(L) := \{p : L \text{ is not } p\text{-stable}\}$. To complete the classification of the 2-regular quaternary lattices, it remains to consider those L for which $|t(L)| \geq 2$. By calculating a stable lattice $\lambda_{(p)}(L)$ for each $\mathbb{Z}$-lattice L classified above, all candidates for $\lambda(L)$ that are stable $\mathbb{Z}$-lattices can be determined. For all $q \neq p$, since there exists an $\epsilon_p \in \mathbb{Z}_p^\times$ such that $\lambda_q(L)_p \cong L_p^{\epsilon_p}$, the local structure L_p is effectively determined for all $p \in t(L)$ by using the above calculation. Using this information, the following result can be established:

Theorem 6.1. *There are exactly* 177 *equivalence classes of primitive even* 2*-regular quaternary* $\mathbb{Z}$*-lattices.*

Remark 6.2. All of the lattices appearing in this list have class number one. Conversely, any lattice of class number one is trivially 2-regular. Consequently, the list obtained constitutes a complete list of primitive even quaternary lattices of class number one. The largest discriminant appearing in the list is $2^4 3^8$. These lattices correspond to the non-classic integral quadratic forms of class number one.

7. n-regular lattices

The finiteness result for 2-regular quaternary lattices has been extended to n-regular lattices for all $n \geq 2$ by Chan and Oh in [**3**]. The treatment in that paper also encompasses almost n-regular and spinor n-regular lattices. The following two theorems summarize the main results obtained.

Theorem 7.1. *For any integer* $n \geq 2$*, there are only finitely many equivalence classes of* n*-regular lattices of rank* $n+3$*.*

THEOREM 7.2. *For any integer $n \geq 2$, there are only finitely many equivalence classes of almost n-regular or spinor n-regular lattices of rank $n+2$.*

When $n \geq 3$, Theorem 7.2 is a consequence of Theorem 7.1 since an almost n-regular lattice is $(n-1)$-regular, and a spinor n-regular lattice of rank $n+2$ is also $(n-1)$-regular; see Lemmas 6.1 and 7.1 of [**3**].

Because of the existence of n-universal lattices for every n, the above two theorems cannot possibly hold for n-regular lattices of large rank. However, it has been shown by Kim, Kim and Oh [**10**] that there are no "new" n-universal lattices with rank beyond some bound N; that is, every n-universal lattice of rank exceeding N must already contain a proper n-universal sublattice (for example, the proof of the Conway-Schneeberger "15-Theorem" shows that there are no new universal lattices of rank exceeding 5). An analogous result will now be stated and proved for "new" n-regular lattices. Some care is required to formulate an appropriate definition, since lattices that are not n-regular can contain n-regular sublattices. So an additional condition on the sets of represented values is imposed.

DEFINITION 7.3. For any $n \geq 2$, an n-regular lattice L is called *new* if L does not contain any proper sublattice M such that $Q_n(L) = Q_n(M)$.

The restriction "$n \geq 2$" is deliberately imposed in the above definition. The case for $n = 1$ requires a somewhat different discussion and will be treated separately in the next section.

LEMMA 7.4. *Let M_p be a $\mathbb{Z}_p$-lattice of rank $\geq n+3$. Then the set*

$$S_n(M_p) = \{Q_n(L_p) : M_p \longrightarrow L_p\}$$

is finite.

PROOF. Let $L_p = \mathfrak{L}_1 \perp \cdots \perp \mathfrak{L}_t$ be a Jordan decomposition of L and s be the first integer such that the rank of $L'_p = \mathfrak{L}_1 \perp \cdots \perp \mathfrak{L}_s$ is at least $n+3$. Since $M_p \longrightarrow L_p$, the scale of $\mathfrak{L}_s$ is bounded and hence the number of possible equivalence classes for L'_p is also bounded. The lemma is then a consequence of Theorems 1 and 3 of [**12**]. □

REMARK 7.5. Suppose M is a lattice of rank $\geq n+3$. Then for any $p \nmid 2dM$, $S_n(M_p)$ has $Q_n(M_p)$ as its only element.

PROPOSITION 7.6. *For any $n \geq 2$, there are only finitely many equivalence classes of new n-regular lattices of a given rank.*

PROOF. Let L be an n-regular lattice of rank m. By Theorem 7.1, we may assume that $m \geq n+4$. Since L is also regular, Corollary 3.2 applies and therefore $\mu_3(L) < C_3$ for some constant C_3. Using Lemmas 3.4 and 3.5 and Section 5 of [**3**], one can show that there exists a constant C_{n+3} such that $\mu_{n+3}(L) < C_{n+3}$.

Suppose that for some $n+3 \leq k < m$, there is a constant C_k such that $\mu_k(L) < C_k$. Let $\mathfrak{T}_k$ be the set of all rank k lattices that are not n-regular and whose k-th successive minima are less than C_k. For each equivalence class of lattices $M \in \mathfrak{T}_k$, select a rank n lattice $\ell^{(1)}(M)$ that is in $Q_n(\text{gen}\, M) \setminus Q_n(M)$. The number of lattices $\ell^{(1)}$ is finite because $\mathfrak{T}_k$ is partitioned into finitely many equivalence classes.

Suppose now that M is a rank k lattice with $\mu_n(M) < C_k$ but $M \notin \mathfrak{T}_k$. Let $\mathfrak{P}$ be the set containing all primes p for which $S_n(M_p) \neq \{Q_n(M_p)\}$. By Remark

7.5, $\mathfrak{P}$ is a finite set. For each $p \in \mathfrak{P}$ and each $\mathcal{Q} \in S_n(M_p) \setminus \{Q_n(M_p)\}$, fix a $\mathbb{Z}_p$-lattice $N_{(p)} \in \mathcal{Q} \setminus Q_n(M_p)$. Choose a $\mathbb{Z}$-lattice ℓ' so that $\ell'_p \cong N_{(p)}$. Then the set $\mathfrak{D} = \{q \neq p : \ell'_q \not\longrightarrow M_q\}$ is finite. For each $q \in \mathfrak{D}$, let $N_{(q)}$ be a lattice on the space spanned by ℓ'_q that is represented by M_q. Such an $N_{(q)}$ always exists because the rank of M is at least $n+3$. Now, let $\ell^{(2)} = \ell^{(2)}(M, p, \mathcal{Q})$ be a $\mathbb{Z}$-lattice such that $\ell^{(2)}_q \cong N_{(q)}$ for all $q \in \mathfrak{D} \cup \{p\}$. This $\mathbb{Z}$-lattice $\ell^{(2)}$ always exists by 81:14 of [**13**]. Note that only finitely many such $\ell^{(2)}$ are constructed in this way.

Let C_{k+1} be the maximum of the n-th successive minima of all the $\ell^{(1)}$ and $\ell^{(2)}$ that have been constructed. Suppose M is a leading $k \times k$ section of L (that is, M is the primitive sublattice of L spanned by the first k vectors in a Minkowski basis for L). If $M \in \mathfrak{T}_k$, then L represents $\ell^{(1)}(M)$ and hence $\mu_{k+1}(L) < C_{k+1}$. Otherwise, there must be a prime p for which $Q_n(M_p) \subsetneq Q_n(L_p)$. In this case, $\ell^{(2)}(M, p, Q_n(L_p))$ is represented by L and, again, $\mu_{k+1}(L) < C_{k+1}$.

A straightforward induction argument shows that $\mu_m(L)$ is bounded by a constant independent of L, and the proposition follows. $\square$

Theorem 7.7. *For any $n \geq 2$, there are only finitely many inequivalent new n-regular lattices.*

Proof. On the contrary, assume that $\{L_i\}$ is an infinite set of inequivalent new n-regular lattices. By Proposition 7.6, it can be assumed that $n+3 \leq \operatorname{rank} L_i < \operatorname{rank} L_{i+1}$ for all $i \geq 1$. As in the proof of Proposition 7.6, $\mu_{n+3}(L_i)$ is bounded by an absolute constant and hence it can be further assumed that all the L_i share a common leading $(n+3) \times (n+3)$ section M. By Lemma 7.4, there exists an infinite set $\{j_1, j_2, \dots\} \subseteq \{1, 2, \dots\}$ such that

$$Q_n(\operatorname{gen}(L_{j_1})) = Q_n(\operatorname{gen}(L_{j_k}))$$

for all $k \geq 1$, which implies that $Q_n(L_{j_1}) = Q_n(L_{j_k})$ for all $k \geq 1$. By Theorem 3.3 of [**10**], there exists a finite subset $\{\ell_1, \dots, \ell_s\}$ of $Q_n(L_{j_1})$ such that every lattice that represents ℓ_i for all $1 \leq i \leq s$ must represent all lattices in $Q_n(L_{j_1})$. Therefore no lattice in $\{L_{j_k}\}$ whose rank is greater than ns can be new. This contradicts the assumption made at the beginning of the proof. $\square$

8. Regular lattices of rank exceeding 4

The finiteness of the number of inequivalent new n-regular lattices no longer holds for $n = 1$ if the definition of new 1-regular lattices is given using the condition in Definition 7.3. The first example of an infinite family of regular quaternary lattices appeared in [**7**]; additional examples of such infinite families can be found in [**9**]. For example, for any positive integer r, the lattice $L^{(r)}$ corresponding to the quadratic form $x^2 + y^2 + z^2 + 2^{2r+1}w^2$ is regular, but $Q(M) \neq Q(L^{(r)})$ for any proper sublattice M of $L^{(r)}$. Indeed, for any quaternary lattice L, it can be seen by comparing the sets of values represented by the respective spaces that no proper sublattice can have the same set of represented values as L. To compensate for this anomaly, the definition of new 1-regular lattice is modified as follows to take into account the representations of the underlying space.

Definition 8.1. A regular lattice L is called *new* if it does not contain any proper sublattice M of rank ≥ 3 for which $Q(M) = Q(L) \cap Q(\mathbb{Q}M)$.

Note that any M in the above definition must be regular. Also, the condition $Q(M) = Q(L) \cap Q(\mathbb{Q}M)$ reduces to $Q(M) = Q(L)$ if $\operatorname{rank} M \geq 4$. The lattices $L^{(r)}$ mentioned in the first paragraph are no longer new according to Definition 8.1, since each contains the proper regular ternary sublattice corresponding to the sum of three squares. The following lemma is the analogue of Lemma 7.4 for the present discussion.

Lemma 8.2. *Let M_p be a $\mathbb{Z}_p$-lattice of rank ≥ 3. Then the set*

$$S(M_p) = \{Q(L_p) \cap Q(\mathbb{Q}_p M_p) : M_p \longrightarrow L_p\}$$

is finite.

Proof. Note that $Q(M_p) \subseteq Q(L_p) \cap Q(\mathbb{Q}_p M_p)$. Moreover, if $a \in Q(L_p) \cap Q(\mathbb{Q}_p M_p)$ and $\operatorname{ord}_p(a)$ is large enough, then $a \in Q(M_p)$. □

Remark 8.3. Let M be a lattice of rank ≥ 3. If $p \nmid 2dM$, then $S(M_p) = \{Q(M_p)\}$.

Theorem 8.4. *There are only finitely many equivalence classes of new regular lattices.*

Proof. As in Section 7, it will first be shown that there are only finitely many equivalence classes of new regular lattices of a given rank $m \geq 4$. If L is such a lattice, then $\mu_3(L)$ is bounded. The proof of Proposition 7.6 will be followed to show inductively that $\mu_k(L)$ is bounded by a constant C_k for any $3 \leq k \leq m$. Suppose that C_k has been found for a particular k. Then define $\mathfrak{T}_k$ and construct $\ell^{(1)}(M)$ for any $M \in \mathfrak{T}_k$ as in the proof of Proposition 7.6.

Suppose M is a lattice of rank k such that $\mu_k(M) < C_k$ but $M \notin \mathfrak{T}_k$. Let $\mathfrak{P}$ be the set of primes p for which $S(M_p)$ contains more than one element. For any $p \in \mathfrak{P}$ and any $\mathcal{Q} \in S(M_p) \setminus \{Q(M_p)\}$, select a $\mathbb{Z}_p$-lattice $N_{(p)} \in \mathcal{Q} \setminus Q(M_p)$ that is on the space $\mathbb{Q}_p M$. This is possible because $\mathcal{Q} \subseteq Q(\mathbb{Q}_p M)$ by definition. Therefore, there exists a $\mathbb{Z}$-lattice ℓ' on the space spanned by M such that $\ell'_p \cong N_{(p)}$. Consider a prime q $(\neq p)$ for which $\ell'_q \not\longrightarrow M_q$. Since ℓ' is on $\mathbb{Q}M$, there exists a $\mathbb{Z}_q$-lattice $N_{(q)}$ on $\mathbb{Q}_q M$ that is represented by M_q. Then proceed to define $\ell^{(2)}(M, p, \mathcal{Q})$, construct the constant C_{k+1}, and confirm that $\mu_{k+1}(M) < C_{k+1}$ as in the proof of Proposition 7.6.

Suppose now that $\{L_1, L_2, \dots\}$ is an infinite set of inequivalent new regular lattices. The above argument implies that $\mu_4(L_i)$ is bounded by an absolute constant, and hence it can be assumed that all the L_i have different ranks and that they share a common leading 4×4 section M. By Lemma 8.2, it can be further assumed that

$$Q(L_1) = Q(L_1) \cap Q(\mathbb{Q}M) = Q(L_i) \cap Q(\mathbb{Q}M) = Q(L_i)$$

for all $i \geq 1$. As in the proof of Theorem 7.7, there exists an integer s such that L_i is not new whenever $\operatorname{rank} L_i > s$. This completes the proof of the theorem. □

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Department of Mathematics, Wesleyan University, Middletown, CT 06459
E-mail address: wkchan@wesleyan.edu

Department of Mathematics, Southern Illinois University Carbondale, Carbondale, IL 62901
E-mail address: aearnest@math.siu.edu

Department of Applied Mathematics, Sejong University, Seoul 143-747, Korea
E-mail address: bkoh@sejong.ac.kr