A FINITENESS THEOREM FOR REPRESENTABILITY OF QUADRATIC FORMS BY FORMS

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Abstract. In this article, Conway-Schneeberger's and Bhargava's results on representability of positive integers by positive definite integral quadratic forms are fully generalized as follows: for any infinite set S of positive definite integral quadratic forms of bounded rank, there is a finite subset S_0 of S such that any positive definite integral quadratic form that represents every element of S_0 represents all elements of S.

1. Introduction

Representation theory of quadratic forms boasts a long and splendid history since Pythagoras. For example, positive integers that are representable by sums of two, three, and four squares were determined by great names like Fermat-Euler, Gauss, and Lagrange, respectively. Hilbert [7] paid a tribute to this fascinating subject by posting two problems among his famous 23 problems for the 20th century - the 11th to quadratic forms over number fields and their integer rings, and the 17th to sums of squares over rational function fields.

Recall Lagrange's *four square theorem*, which states: *the integral quadratic form* $x^2+y^2+z^2+u^2$ *represents all positive integers.* This celebrated statement had been generalized in many different directions such as Waring's problem and Pythagoras numbers, to name a few. One interesting generalization was made by Ramanujan [26] in the early 20th century, who found and listed all 55 positive definite integral quaternary diagonal quadratic forms, up to equivalence, that represent all positive integers. Dickson [4] called such forms *universal* and confirmed Ramanujan's list except one form from the list that was not universal. Later, Willerding [28] added 124 quaternary non-diagonal universal forms to the list, up to equivalence, and claimed that the list is complete. It is not hard to show that 'quaternary' is the best possible in the sense that there is no ternary universal form. Throughout this paper, by *integral* quadratic forms we mean *classic* ones, that is, quadratic forms with integer coefficients such that coefficients of non-diagonal terms are multiples of 2.

In 1930, Mordell [20] proved the *five square theorem*, which states: *the integral quadratic form* $x^2+y^2+z^2+u^2+v^2$ *represents all positive definite integral binary quadratic forms.* (There is a very interesting new direction of generalizing Lagrange's four square theorem, called the quadratic Waring's problem [21], [12]. See [14-17] for recent development in this direction.) Such a form is called *2-universal.*

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Recently, all quinary 2-universal forms were determined by the authors [13]. For positive integers $k, 3 \le k \le 10$, k-universal forms were also investigated in [13] and [22]. In 1941, Maass [18] proved the *three square theorem*, which states: *the integral quadratic form $x^2+y^2+z^2$ is universal over $\mathbb{Q}(\sqrt{5}\,)$, i.e., it represents all totally positive integers over the ring of integers of $\mathbb{Q}(\sqrt{5}\,)$.* All positive definite integral ternary universal forms over real quadratic fields were determined in [3]. For further development on universal forms over number fields, see [9-11] and [6].

In 1997, Conway and Schneeberger announced so called the *fifteen theorem*, which characterizes the (1-)universality by the representability of a finite set of numbers, namely, 1, 2, 3, 5, 6, 7, 10, 14, and 15 (see [27]). Using this criterion, they corrected several mistakes in the Willerding's list and announced the new and complete list of the 204 quaternary universal forms, up to equivalence. This result was so stunning and beautiful that it was introduced in the Notices of American Mathematical Society [5]. It shed new light in the global theory of representations of quadratic forms. Motivated by the 15-Theorem, the authors proved 2-universal and 8-universal analogies in [13] and [22]. Then came Bhargava's generalization [2]. It was announced that he proved: for any infinite set S of positive integers there is a finite subset S_0 of S such that any positive definite integral quadratic form that represents every element of S_0 represents all elements of S. As a byproduct, he found S_0 for some interesting sets S, for example, the set of all primes, the set of all positive odd integers, and so on.[1]

In this article, we generalize Conway-Schneeberger's and Bhargava's results as follows: *for any infinite set S of positive definite integral quadratic forms of bounded rank, there is a finite subset S_0 of S such that any positive definite integral quadratic form that represents every element of S_0 represents all of S.*

We adopt lattice theoretic language. A $\mathbb{Z}$-*lattice* L is a finitely generated free $\mathbb{Z}$-module equipped with a non-degenerate symmetric bilinear form B such that $B(L,L) \subseteq \mathbb{Z}$. The corresponding quadratic map is denoted by Q. Note that $\mathbb{Z}$-lattices naturally correspond to integral quadratic forms.

For a $\mathbb{Z}$-lattice $L = \mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_n$ where $\{\,\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\,\}$ is a fixed basis, we write

$$L \cong (B(\mathbf{x}_i, \mathbf{x}_j)).$$

The right hand side matrix is called a *matrix presentation* of L. For $\mathbb{Z}$-sublattices L_1, L_2 of L, we write $L = L_1 \perp L_2$ when $L = L_1 \oplus L_2$ and $B(\mathbf{v}_1, \mathbf{v}_2) = 0$ for all $\mathbf{v}_1 \in L_1, \mathbf{v}_2 \in L_2$. If L admits an orthogonal basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, we call L *diagonal* and simply write

$$L \cong \langle Q(\mathbf{x}_1), Q(\mathbf{x}_2), \ldots, Q(\mathbf{x}_n)\rangle.$$

We call L *non-diagonal* otherwise. L is called *positive definite* or simply *positive* if $Q(\mathbf{v}) > 0$ for any $\mathbf{v} \in L, \mathbf{v} \neq \mathbf{0}$. We call $Q(\mathbf{v})$ the *norm* of $\mathbf{v}$. In this article, we assume that

Every $\mathbb{Z}$-lattice is positive definite

unless stated otherwise. The ideal of $\mathbb{Z}$ generated by $B(L,L)$ is called the *scale* of L, denoted by $\mathfrak{s}(L)$. As usual, $dL = \det(B(\mathbf{x}_i, \mathbf{x}_j))$ is called the *discriminant* of

[1] In recent private communication with Bhargava, the authors learned that he also obtained the same result as ours. But unfortunately we haven't seen his proof yet.

L. For any $\mathbb{Z}$-lattice (or $\mathbb{Z}_p$-lattice) L and $m \in \mathbb{Z}$ (or $m \in \mathbb{Z}_p$) , $\sqrt{m}L$ denotes the $\mathbb{Z}$-lattice ($\mathbb{Z}_p$-lattice, respectively) obtained from scaling L by m.

We define $RL := R \otimes L$ for any ring R containing $\mathbb{Z}$. If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an orthogonal basis of the quadratic space $V = \mathbb{Q}L$ or $\mathbb{Q}_pL$, we write

$$V \cong (Q(\mathbf{x}_1), Q(\mathbf{x}_2), \ldots, Q(\mathbf{x}_n))$$

for convenience.

Let ℓ, L be $\mathbb{Z}$-lattices. We say that L *represents* ℓ and write $\ell \to L$ if there is an injective $\mathbb{Z}$-linear map σ from ℓ into L preserving norms. Such σ is called a *representation.*

A positive $\mathbb{Z}$-lattice L is called *n-universal* if L represents all n-ary positive $\mathbb{Z}$-lattices ℓ. So, Lagrange's four square theorem is precisely the 1-universality of $I_4 \cong \langle 1, 1, 1, 1\rangle$ and each of Ramanujan's forms above corresponds to a 1-universal quaternary diagonal $\mathbb{Z}$-lattice. Furthermore, Mordell's five square theorem is nothing but the 2-universality of I_5. It is well known that I_n is $(n-3)$-universal if $4 \le n \le 8$ (see [13] for example). For a given set S of $\mathbb{Z}$-lattices, a $\mathbb{Z}$-lattice L is called *S-universal* if L represents all $\mathbb{Z}$-lattices in S. For any unexplained terminology and basic facts about $\mathbb{Z}$-lattices, we refer the readers to O'Meara's book [23].

2. Lemmas

Let $S = \{\ell_1, \ell_2, \ldots, \ell_t, \ldots\}$ be an infinite set of $\mathbb{Z}$-lattices of rank n. For each $\ell \in S$, we fix a Minkowski reduced basis $\{\mathbf{x}_1(\ell), \mathbf{x}_2(\ell), \ldots, \mathbf{x}_n(\ell)\}$ of ℓ. We define

$$\ell(i) := \mathbb{Z}\mathbf{x}_i(\ell) + \mathbb{Z}\mathbf{x}_{i+1}(\ell) + \cdots + \mathbb{Z}\mathbf{x}_n(\ell)$$

for $i = 1, 2, \ldots, n$. Note that $\{\mathbf{x}_i(\ell), \mathbf{x}_{i+1}(\ell), \ldots, \mathbf{x}_n(\ell)\}$ is also a Minkowski reduced basis of $\ell(i)$ and

$$|B(\mathbf{x}_i(\ell), \mathbf{x}_j(\ell))| \le B(\mathbf{x}_i(\ell), \mathbf{x}_i(\ell)) = \min(\ell(i))$$

for all $i \le j$. For any $\mathbb{Z}$-lattice L, we define ℓ_L as follows:

$$\ell_L := \begin{cases} \ell_j & \text{if } \ell_i \to L \text{ for all } i < j \text{ and } \ell_j \not\to L, \\ 0 & \text{if such an } j \text{ does not exist.} \end{cases}$$

We inductively define T_i's as follows: Let T_1 be the set of all $\mathbb{Z}$-lattices of rank n that represent ℓ_1. For $i \ge 1$, let

$$U_i := \{L \in T_i \mid \ell_L \ne 0\},$$

and for each $L \in U_i$ let $U(L)$ be the set of $\mathbb{Z}$-lattices M satisfying the following two conditions:

(1) M represents L and ℓ_L,
(2) No sublattice of M of rank less than $\text{rank}(M)$ can represent both L and ℓ_L.

Then define

$$T_{i+1} := \bigcup_{L \in U_i} U(L).$$

Note that $\text{rank}(M) \le n + \text{rank}(L)$.

Lemma 2.1. *Let $S(p)$ be an infinite set of $\mathbb{Z}_p$-lattices of rank n. Then there exists a finite subset $S_0(p)$ of $S(p)$ such that any $\mathbb{Z}_p$-lattice that represents every element of $S_0(p)$ represents all of $S(p)$.*

Proof. The proof is almost identical to that of Lemma 1.5 of [8]. □

Lemma 2.2. *Let L be a $\mathbb{Z}_p$-lattice of rank $N \geq 4n+12$ and m be any integer not less than $m_0(L) := ord_p(\mathfrak{s}(L_k))$, where L_k is the last component of a Jordan decomposition of L. Then for any $\alpha \in \mathbb{Z}_p$, $\sqrt{\alpha p^m} I_{n+3} \to K$ over $\mathbb{Z}_p$ for any primitive sublattice K of L with rank$(K) \geq N-n$.*

Proof. Let $K = K_0 \perp K_1 \perp \cdots \perp K_t$ be a Jordan decomposition of K with $\mathfrak{s}(K_i) = p^i\mathbb{Z}_p$ or $K_i = 0$. Let

$$K_- := K_0 \perp K_1 \perp \cdots \perp K_{m_0}, \quad K_+ := K_{m_0+1} \perp K_{m_0+2} \perp \cdots \perp K_t,$$

and $r = \text{rank}(K_+)$. Since K is a primitive sublattice of L, we may choose a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ of L extending a basis of K. The rank of the matrix $(B(\mathbf{x}_i, \mathbf{x}_j))$ over $\mathbb{Z}/p^{m_0+1}\mathbb{Z}$ is exactly N. From this we obtain $2r \leq N$. Therefore rank$(K_-) \geq n+6$, which implies the lemma by Theorem 1 and Theorem 3 of [24]. □

Lemma 2.3. *Let L be a $\mathbb{Z}$-lattice of rank $N \geq 4n+12$. For any $\mathbb{Z}$-lattice ℓ of rank n and M representing both ℓ and L, there is an integer q depending only on L for which $\sqrt{q} I_{n+3} \to \sigma(\ell)^\perp(M)$ over $\mathbb{Z}_p$ for every prime p, where $\sigma : \ell \to M$ is the representation and $\sigma(\ell)^\perp(M)$ is the orthogonal complement of $\sigma(\ell)$ in M.*

Proof. Without loss of generality, we may assume that $\ell, L \subset M$. Let

$$\ell = \mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_n,$$

and write $\mathbf{x}_i = \mathbf{y}_i + \mathbf{z}_i$, where $\mathbf{y}_i \in \mathbb{Q}L$ and $\mathbf{z}_i \in \mathbb{Q}L^\perp(M)$. Choose an integer d such that $d\mathbf{y}_i \in L$ for all i and define

$$\tilde{\ell} := \mathbb{Z}d\mathbf{y}_1 + \cdots + \mathbb{Z}d\mathbf{y}_n \subset L.$$

Then $\tilde{\ell}^\perp(L)$ is a primitive sublattice of L and $\tilde{\ell}^\perp(L) \subset \ell^\perp(M)$. Let

$$q = q(L) := \prod_{p\,:\,\text{prime}} p^{m_0(p)},$$

where $m_0(p)$ is the p-order of the scale of the last component of a Jordan decomposition of L_p for each prime p. Since $m_0(p) = 0$ for almost all primes p, q is well defined. The lemma then follows from Lemma 2.2. □

Lemma 2.4. *The set T_i is finite for every $i = 1, 2, 3, \ldots$. Furthermore, every $\mathbb{Z}$-lattice in T_i represents ℓ_j for all $j \leq i$.*

Proof. For any $\mathbb{Z}$-lattice A of rank a, denote the a-th, i.e., the last, successive minimum of A by $\overline{\mu}(A)$. T_1 is clearly finite. For $i \geq 2$, let $L \in U_{i-1}$ and $M \in U(L)$. Suppose that $\overline{\mu}(M) > \max(\overline{\mu}(\ell_L), \overline{\mu}(L))$. Let K be the sublattice of M generated by the vectors $\mathbf{x} \in M$ with $Q(\mathbf{x}) < \overline{\mu}(M)$ and $\widetilde{K} := \mathbb{Q}K \cap M$. Then $\widetilde{K}$ is a sublattice with rank less than rank(M) and represents both L and ℓ_L, which is a contradiction. So the last successive minimum of M is bounded above. From this follows the first assertion. The second assertion is trivial from the definition of $U(L)$. □

3. Main Results and Applications

We are now ready to prove our main results.

Proposition 3.1. *Assume that there exists an r such that every $\mathbb{Z}$-lattice in T_r is S-universal. Let*

$$S_0 := \{\ell_1\} \cup \left\{ \ell_L \,:\, L \in \bigcup_{i=1}^{r-1} U_i \right\}.$$

Then every S_0-universal $\mathbb{Z}$-lattice is S-universal.

Proof. Let M be any $\mathbb{Z}$-lattice that represents all $\mathbb{Z}$-lattices in S_0. If $K \to M$ for some $K \in \bigcup_{i=1}^{r-1}(T_i - U_i)$, then M is S-universal. Therefore we may assume that $K \not\to M$ for any $K \in \bigcup_{i=1}^{r-1}(T_i - U_i)$. Then for all $1 \le j \le r-1$, we can find $\mathbb{Z}$-sublattices M_j of M such that $M_j \in U_j$ and $M_j \subseteq M_{j+1}$ $(1 \le j \le r-2)$. Since $M_{r-1} \in U_{r-1}$ and $\ell_{M_{r-1}} \to M$, there exists at least one $\mathbb{Z}$-lattice in T_r that is represented by M. $\square$

Proposition 3.2. *There is an integer r satisfying the condition of Proposition 3.1.*

Proof. Suppose that such an r does not exist. Let s be the smallest integer such that the ranks of all lattices in T_s are greater than $n+3$ and P be the finite set of all primes p for which L_p is not unimodular for some $L \in T_s$. By applying Lemma 2.1 for each prime in P, we may choose an integer $s_1 \ge s$ such that every $\mathbb{Z}$-lattice $L \in T_{s_1}$ represents all $\ell \in S$ over $\mathbb{Z}_p$ for every prime p. Without loss of generality, we may further assume that $\mathrm{rank}(L) \ge 4n+12$ for all $L \in T_{s_1}$. Then by Theorem 2.1 of [8], there exists an integer c_1 such that every $\mathbb{Z}$-lattice $M \in T_{s_1}$ represents all $\ell \in S$ provided that $B(\mathbf{x}_1(\ell), \mathbf{x}_1(\ell)) = \min(\ell) \ge c_1$. For each $\mathbb{Z}$-lattice $L \in T_{s_1}$, let $q(L)$ be the integer satisfying Lemma 2.3 and let

$$q := \prod_{L \in T_{s_1}} q(L).$$

Let $1 \le i \le n-1$. Assume that there exist $c_i \ge c_1$ and s_i such that every $\mathbb{Z}$-lattices in T_{s_i} represents all $\ell \in S$ provided that $B(\mathbf{x}_i(\ell), \mathbf{x}_i(\ell)) = \min \ell(i) \ge c_i$. So, if we let S_i be the subset of S consisting of all $\mathbb{Z}$-lattices that are not represented by some $\mathbb{Z}$-lattices in T_{s_i}, then for every $\ell \in S_i$, $|B(\mathbf{x}_j(\ell), \mathbf{x}_k(\ell))| \le c_i$ for all $1 \le j \le i$, $1 \le k \le n$. Let $M_{i\times n}$ be the set of all $i \times n$ matrices (c_{jk}) with $|c_{jk}| \le c_i$. For $C = (c_{jk}) \in M_{i\times n}$, let $S_i(C)$ be the subset of S_i consisting of all $\mathbb{Z}$-lattices ℓ satisfying $B(\mathbf{x}_j(\ell), \mathbf{x}_k(\ell)) = c_{jk}$ for all $1 \le j \le i$, $1 \le k \le n$. For two $\ell, \ell' \in S_i(C)$, define $\ell \sim \ell'$ by

$$B(\mathbf{x}_j(\ell), \mathbf{x}_k(\ell)) \equiv B(\mathbf{x}_j(\ell'), \mathbf{x}_k(\ell')) \pmod{q}.$$

for all $i+1 \le j \le n$, $i+1 \le k \le n$. This is an equivalence relation. If we let $S_i^*(C)$ be a complete set of representatives for the equivalence classes in $S_i(C)$, then $S_i^*(C)$ is a finite subset of $S_i(C)$. Define

$$W_i := \bigcup_{C \in M_{i\times n}} S_i^*(C)$$

and let $s_{i+1} \ge s_i$ be an integer such that every $\mathbb{Z}$-lattice in $T_{s_{i+1}}$ represents all $\mathbb{Z}$-lattices in W_i. Such an integer always exists by Lemma 2.4. Then there exists an integer $c_{i+1} \geq c_i$ such that for every $L \in T_{s_{i+1}}$, L represents all $\mathbb{Z}$-lattices $\ell \in S$ provided that

$$\min(\ell(i+1)) = B(\mathbf{x}_{i+1}(\ell), \mathbf{x}_{i+1}(\ell)) \geq c_{i+1}$$

by Theorem 2.1 of [8] and Lemma 2.3. In this manner, we obtain c_n and s_n so that every $\mathbb{Z}$-lattice in T_{s_n} represents all $\mathbb{Z}$-lattices $\ell \in S$ provided that

$$\min(\ell(n)) = \overline{\mu}(\ell) \geq c_n,$$

which implies that every $\mathbb{Z}$-lattice in T_{s_n} represents almost all $\mathbb{Z}$-lattices of S. This is a contradiction. □

Theorem 3.3. *Let S be an infinite set of $\mathbb{Z}$-lattices of a given rank. Then there exists a finite subset S_0 of S such that every S_0-universal $\mathbb{Z}$-lattice is S-universal.*

Proof. The theorem follows immediately from Propositions 3.1 and 3.2. □

An S-universal $\mathbb{Z}$-lattice L is said to be *new* if it does not contain a sublattice of lower rank that is also S-universal and *old*, otherwise.

Corollary 3.4. *Let S be an infinite set of $\mathbb{Z}$-lattices of rank n. Then there exist positive integers $N_1(S), N_2(S)$ such that*
(1) *there is no S-universal $\mathbb{Z}$-lattice of rank less than $N_1(S)$ and*
(2) *there is no new S-universal $\mathbb{Z}$-lattice of rank greater than $N_2(S)$.*
In particular, the total number of isometry classes of new S-universal $\mathbb{Z}$-lattices is finite.

Proof. This follows immediately from the construction of T_i's and the proof of Proposition 3.2. □

Let $\mathcal{P}_n$ denote the set of all $\mathbb{Z}$-lattices of rank n. It is known that $N_1(\mathcal{P}_1) = 4$, $N_2(\mathcal{P}_1) = 5$ and $N_1(\mathcal{P}_2) = N_2(\mathcal{P}_2) = 5$ (see [1], [13]). The values of $N_1(\mathcal{P}_n)$ are known ([13], [22]) for small n's, for example, $N_1(\mathcal{P}_n) = n + 3$ for $1 \leq n \leq 5$, and

$$N_1(\mathcal{P}_6) = 13,\ N_1(\mathcal{P}_7) = 15,\ N_1(\mathcal{P}_8) = 16,\ N_1(\mathcal{P}_9) = 28,\ N_1(\mathcal{P}_{10}) = 30.$$

Define

$$\Gamma(S) := \{\, S_0 \subset S \,:\, \text{every } S_0\text{-universal } \mathbb{Z}\text{-lattice is } S\text{-universal} \,\}.$$

The fifteen theorem asserts that $S_0 = \{\, 1, 2, 3, 5, 6, 7, 10, 14, 15 \,\}$ is a unique minimal set in $\Gamma(\mathcal{P}_1)$. The uniqueness of a minimal S_0, however, is not guaranteed in general when $n \neq 1$. For example, if we let $S = \{\, \langle 2^i, 2^j, 2^k \rangle \,:\, i, j, k \geq 0 \,\}$, then

$$S_0 = \{\, \langle 1,1,1 \rangle, \langle 1,1,2 \rangle \,\} \text{ and } S_0' = \{\, \langle 1,1,1 \rangle, \langle 2,2,2 \rangle \,\}$$

are both minimal in $\Gamma(S)$. The following two questions seem to be quite interesting but difficult :

(1) *For which S is there a unique minimal $S_0 \in \Gamma(S)$?*
(2) *Is $|S_0| = \gamma(S)$ for every minimal $S_0 \in \Gamma(S)$? If not, when ?*

where $\gamma(S) := \min\{\, |S_0| \,:\, S_0 \in \Gamma(S) \,\}$, which is finite by Theorem 3.3.

Remark. The authors expect Theorem 3.3 and Corollary 3.4 hold for totally positive $\mathcal{O}_F$-lattices, where F is any totally real number field because all ingredients (reduction theory, local and global theory, and Hsia-Kitaoka-Kneser theorem, *etc*...) in the proofs are available over totally real number fields.

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