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REGULARITY OF MAPPINGS OF G -STRUCTURES OF FROBENIUS TYPE

CHONG KYU HAN

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ABSTRACT. A notion of Frobenius type for a G -structure is defined. It is shown that a mapping f between C^∞ (resp. C^ω) manifolds with a G -structure of the Frobenius type is C^∞ (resp. C^ω) if $f \in C^k$, where the integer k depends on the order of the Frobenius type. It is also shown that a G -structure of finite order is of the Frobenius type.

0. INTRODUCTION

Let G be a Lie subgroup of $Gl(n; \mathbf{R})$. A G -structure on a C^∞ manifold M of dimension n is a C^∞ sub-bundle P of the bundle of linear frames over M with structure group G . Let f be a C^1 diffeomorphism of M with a G -structure P onto another manifold \tilde{M} with a G -structure \tilde{P} . f is called a G -mapping if for any frame field (e_1, \dots, e_n) over M belonging to P , (f_*e_1, \dots, f_*e_n) is a frame field over \tilde{M} belonging to \tilde{P} . We are concerning with regularity of G -mappings. Locally, the above condition is a system of partial differential equations of order 1 and the question of regularity naturally arises. Our approach is to reduce the regularity problem of G -mappings to that of infinitesimal automorphisms of P and then to find conditions on P which imply the regularity of infinitesimal automorphisms. In cases, regularity of infinitesimal automorphisms of a G -structure can be deduced by properties of G only. A well-known example is that if the associated Lie algebra of G contains no matrix of rank 1 then an infinitesimal automorphism of a G -structure satisfies a system of elliptic linear partial differential equations with C^∞ coefficients, and therefore is C^∞ (see the proof of Theorem 4.1 of [5]). But in general, we need conditions on a specific G -structure P in addition to the conditions on G . This paper concerns the cases where regularity of G -mappings follows from the Frobenius theorem.

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Our viewpoint is purely local, so, for example, a “function” must be understood as a germ of a function at a reference point and in §1 we work on an open set $\mathcal{O} \subseteq \mathbf{R}^n$ instead of a manifold M . Often, we have to think of open subsets of \mathcal{O} , however, we denote them also by \mathcal{O} , for there is no danger of confusion. If the underlying manifolds and G -structures on them are real analytic we get a real analytic version of this paper by changing every C^∞ to C^ω .

1. MAPPINGS OF G -STRUCTURES OF FROBENIUS TYPE

Let \mathcal{O} be an open subset of \mathbf{R}^n and $f = (f^1, \dots, f^l)$ be a system of real valued functions of \mathcal{O} . f is said to satisfy a complete system of order m if every partial derivative of f^j , $j = 1, \dots, l$, of order m can be expressed as a C^∞ function of the partial derivatives of (f^1, \dots, f^l) of order $< m$: For each $j = 1, \dots, l$, each multi-index α with $|\alpha| = m$,

$$D^\alpha f^j = H_\alpha^j(x, D^\beta f: |\beta| < m), \quad H_\alpha^j \in C^\infty$$

where $D^\beta f \equiv (D^\beta f^1, \dots, D^\beta f^l)$.

We will use summation convention in this section: repeated indices denote the summation over $1, \dots, n$. A complete system for a vector field $X = \xi^i(\partial/\partial x_i)$ on \mathcal{O} is a complete system for its components (ξ^1, \dots, ξ^n) . If \mathcal{O} is equipped with a G -structure P let (e_1, \dots, e_n) be a frame belonging to P and set $L_X e_j = a_j^i e_i$, where L is the Lie derivative. Then a C^1 vector field X is an infinitesimal automorphism of P if and only if the matrix $[a_j^i]$ belongs to \mathcal{G} , the associated Lie algebra of G (cf. [5]). This condition can be expressed as a system of linear partial differential equations of first order: Let $X = u^i e_i$ and let

$$[e_i, e_j] = b_{ij}^k e_k.$$

Then

$$\begin{aligned} L_X e_j &= [u^i e_i, e_j] \\ &= (-e_j u^i + u^k b_{kj}^i) e_i, \end{aligned}$$

so we have

$$a_j^i = -e_j u^i + u^k b_{kj}^i.$$

Since \mathcal{G} is a linear subspace of $gl(n; \mathbf{R})$, it is defined by linear equations, namely,

$$\mathcal{G} = \{y_j^i \in \mathbf{R}^{n^2} : c_{i\lambda}^j y_j^i = 0, \quad \lambda = 1, \dots, N\},$$

where N is the codimension of \mathcal{G} in $gl(n; \mathbf{R})$ and c are constants. Thus we get

$$c_{i\lambda}^j (-e_j u^i + u^k b_{kj}^i) = 0, \quad \lambda = 1, \dots, N.$$

To express the above in terms of local coordinates, let $X = \xi^i(\partial/\partial x_i)$ and let $(\partial/\partial x_j) = b_j^i e_i$, then $u^i = b_j^i \xi^j$ and the above equation becomes

$$(1.1) \quad c_{i\lambda}^j [-e_j (b_k^i \xi^k) + b_t^k b_{kj}^i \xi^t] = 0, \quad \lambda = 1, \dots, N.$$

(1.1) is a system of linear PDE of first order for $\xi = (\xi^1, \dots, \xi^n)$ with C^∞ coefficients. By a prolongation of (1.1) we shall mean a system of linear PDE obtained from (1.1) through a process of repeated differentiations, additions and multiplications by C^∞ functions. The main result of this paper is the following.

Theorem 1.1. *Let G be a Lie subgroup of $Gl(n; \mathbf{R})$. Suppose that \mathcal{O} and $\tilde{\mathcal{O}}$ are open neighborhoods of the origin of \mathbf{R}^n with G -structure P and \tilde{P} , respectively. Let $f: \mathcal{O} \rightarrow \tilde{\mathcal{O}}$ be a G -mapping of class C^k for some sufficiently large k . Suppose that the equation (1.1) for the infinitesimal automorphisms of P has a prolongation to a complete system of order m and that \tilde{P} has the same property. Then f satisfies a complete system of order $m + 1$.*

Note that the existence of an infinitesimal automorphism on \mathcal{O} or $\tilde{\mathcal{O}}$ is not assumed. In general, if a system of functions $f = (f^1, \dots, f^l)$ satisfies a complete system of order m , then by introducing new variables for the partial derivatives of order $< m$, we can construct a Pfaffian system on a submanifold of \mathbf{R}^N where $N = n +$ number of the newly introduced variables, so that f may be identified with an integral manifold as in the proof of Theorem 1.1. So the questions of existence, regularity and uniqueness of f reduce to the Frobenius theorem. We will call G -structures as in Theorem 1.1 Frobenius type.

Definition 1.2. *A G -structure P on a C^∞ manifold M is a Frobenius type of order m if the equation (1.1) for the infinitesimal automorphisms of P in terms of any local coordinates has a prolongation to a complete system of order m .*

We have

Corollary 1.3. *Let M and \tilde{M} be C^∞ manifolds with G -structure P and \tilde{P} , respectively, and $f: M \rightarrow \tilde{M}$ be a G -mapping. If P and \tilde{P} are Frobenius type of order m and $f \in C^{m+1}$ then $f \in C^\infty$ and f is locally determined by $\{D^\beta f(0): |\beta| \leq m\}$ at any point $0 \in M$.*

Now let P be a G -structure of Frobenius type of order m on an open set $\mathcal{O} \subseteq \mathbf{R}^n$. Let

$$(1.2) \quad D^\alpha \xi^i = H_\alpha^i(x, D^\beta \xi: |\beta| < m), \quad \forall \alpha \text{ with } |\alpha| = m, \forall i = 1, \dots, n,$$

be a prolongation of (1.1) to a complete system. Note that each H_α^i is linear in $D^\beta \xi$. To realize the process of prolongation in the jet spaces, we introduce new variables

$$p_\beta = (p_\beta^1, \dots, p_\beta^n) \text{ for } D^\beta \xi = (D^\beta \xi^1, \dots, D^\beta \xi^n).$$

For each positive integer k , the k th order jet space is $J^k \equiv \mathcal{O} \times \mathbf{R}^{(k)} = \{(x, \xi, p)\}$, where $p = (p_\beta: |\beta| \leq k)$ and (k) is the number of the variables (ξ, p) . In J^k consider the submanifold Δ^k which is defined by (1.1) and all

the equations obtained by differentiating (1.1) in all possible ways up to order $k-1$. Let $X = \xi^i(\partial/\partial x_i)$ be a C^k vector field. Then the k -graph of X is the submanifold of J^k

$$j_X^k(x) \equiv (x, \xi(x), D^\beta \xi(x) : |\beta| \leq k)$$

and k th order contact systems is

$$\Omega^k \equiv \{w \in T^*J^k : (j_X^k)^*w = 0, \text{ for all vector fields } X \text{ on } \mathcal{O}\}.$$

Then Ω^k is spanned by

$$w^k \equiv d\xi^k - p_j^i dx^j \text{ and}$$

$$w_\beta^i \equiv dp_\beta^i - p_{(\beta,j)}^i dx^j, \text{ where}$$

(β, j) denotes the multi-index $(\beta_1, \dots, \beta_j + 1, \dots, \beta_n)$ if $\beta = (\beta_1, \dots, \beta_n)$, (cf. [8 and 9]). We prove

Lemma 1.4. *Let P be a G -structures on \mathcal{O} and let J^k , Δ^k and Ω^k be as above. Then the following are equivalent:*

- (i) P is of Frobenius type of order m ;
- (ii) the $(m-1)$ th contact system Ω^{m-1} defines an n -dimensional distribution \mathcal{D} on Δ^{m-1} , with $dx^1 \wedge \dots \wedge dx^n \neq 0$ on each integral element of \mathcal{D} .

Proof of Lemma 1.4. (i) \Rightarrow (ii): Let (1.2) be a prolongation of (1.1) to a complete system. (1.2) is equivalent to the total differential equation

$$d(D^\beta \xi^i) = H_{(\beta,j)}^i(x, D^\gamma \xi : |\gamma| \leq m-1) dx^j, \quad \forall \beta \text{ with } |\beta| = m-1,$$

$\forall i = 1, \dots, n$. This implies that on Δ^{m-1}

$$\Omega_\beta^i \equiv dp_\beta^i - H_{(\beta,j)}^i(x, \xi, p_\gamma) dx^j = 0,$$

$\forall \beta$ with $|\beta| = m-1$, $\forall i = 1, \dots, n$. Let \mathcal{D} be the distribution on Δ^{m-1} given by $\Omega^{(m-1)} = 0$. Then on each integral element of \mathcal{D} we have

$$d\xi^i = p_j^i dx^j,$$

$$dp_\beta^i = p_{(\beta,j)}^i dx^j, \quad \forall \beta \text{ with } |\beta| < m-1,$$

$$dp_\beta^i = H_{(\beta,j)}^i(x, \xi, p) dx^j, \quad \forall \beta \text{ with } |\beta| = m-1, \text{ and}$$

$$dx^1 \wedge \dots \wedge dx^n \neq 0.$$

Therefore, \mathcal{D} is an n -dimensional distribution.

(ii) \Rightarrow (i): Let \mathcal{D} be the distribution as in (ii). Let $\varphi^1, \dots, \varphi^\nu$ be differential 1-forms on Δ^{m-1} which generate the differential ideal associated with \mathcal{D} , where $\nu = (\text{dimension of } \Delta^{m-1}) - n$. Set

$$\varphi^j = a^j dx + b^j d\xi + c^j dp, \quad j = 1, \dots, \nu,$$

where a^j , b^j and c^j are row vectors and dx , $d\xi$ and dp are column vectors so that $a^j dx = a_1^j dx^1 + \dots + a_n^j dx^n$, etc. Since each integral element of \mathcal{D} is n -dimensional subspace of $T(\Delta^{m-1})$ on which $dx^1 \wedge \dots \wedge dx^n \neq 0$, we can solve $\varphi^j = 0$, $j = 1, \dots, \nu$, for $d\xi$ and dp we get

$$\begin{cases} d\xi^i = h_j^i dx^j \\ dp_\beta^i = h_{(\beta,j)}^i dx^j, \quad \forall \beta \text{ with } |\beta| \leq m-1, \quad \forall i = 1, \dots, n, \end{cases}$$

where h are C^∞ functions on Δ^{m-1} .

This implies that if $\xi = (\xi^1, \dots, \xi^n)$ satisfies (1.1) then

$$d(D^\beta \xi^i) - h_{(\beta,j)}^i(x, D^\gamma \xi: |\gamma| \leq m-1) dx^j = 0,$$

for all β with $|\beta| \leq m-1$. In particular, if $|\beta| = m-1$, the above equation is equivalent to

$$D^{(\beta,j)} \xi^i = h_{(\beta,j)}^i(x, D^\gamma \xi: |\gamma| \leq m-1), \quad |\beta| = m-1,$$

which is a complete system of order m . Q.E.D.

Proof of Theorem 1.1. Let (1.2) be the complete system for the infinitesimal automorphism of P and let Ω^k be the k th contact system on $\Delta^k \subseteq J^k$ for each $k = 1, \dots, m-1$. For each multi-index β with $|\beta| = m-1$ and each $i = 1, \dots, n$, let $\Omega_\beta^i \equiv dp_\beta^i - H_{(\beta,j)}^i dx^j$, where H are the same as in the complete system (1.2). Let \mathcal{D} be the distribution as in Lemma 1.4. We will put tilde on the corresponding notions on $\tilde{\mathcal{O}}: \tilde{J}^{m-1} = \tilde{\mathcal{O}} \times \mathbf{R}^{(m-1)} = (\tilde{x}, \tilde{\xi}, \tilde{p})$, etc. A C^{m+1} diffeomorphism $f: \mathcal{O} \rightarrow \tilde{\mathcal{O}}$ naturally defines a C^1 diffeomorphism $F^k: J^k \rightarrow \tilde{J}^k$ for each $k = 1, \dots, m-1$ as follows: Let $F^k(x, \xi, p) = (\tilde{x}, \tilde{\xi}, \tilde{p})$.

Then

$$(1.3) \quad \tilde{x}^i(x, \xi, p) = f^i(x)$$

$$(1.4) \quad \tilde{\xi}^i(x, \xi, p) = \xi^\lambda \frac{\partial f^i}{\partial x^\lambda}, \quad i = 1, \dots, n$$

and define $\tilde{P}(x, \xi, p)$ by chain rule, namely

$$(1.5) \quad \begin{aligned} \tilde{p}_j^i(x, \xi, p) &= \frac{\partial \tilde{\xi}^i}{\partial \tilde{x}^k} \\ &= \frac{\partial \tilde{\xi}^i}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^j} \end{aligned}$$

substitute (1.4) for ξ^i and p_μ^λ for $\frac{\partial \xi^\lambda}{\partial x^\mu}$

$$= \left(p_\mu^\lambda \frac{\partial f^i}{\partial x^\lambda} + \xi^\lambda \frac{\partial^2 f^i}{\partial x^\mu \partial x^\lambda} \right) \frac{\partial x^\mu}{\partial \tilde{x}^j}$$

each $\frac{\partial x^\mu}{\partial \tilde{x}^j}$ is an entry of $\left[\frac{\partial f^i}{\partial x^j} \right]_{i,j=1,\dots,n}^{-1}$,

therefore a C^∞ function of $\frac{\partial f^i}{\partial x^j}$, $i, j = 1, \dots, n$, so

$$= \xi^\lambda \frac{\partial^2 f^i}{\partial x^\mu \partial x^\lambda} \frac{\partial x^\mu}{\partial \tilde{x}^j} + a_\lambda^\mu p_\mu^\lambda,$$

where a_λ^μ are C^∞ functions of $(D^\gamma f: |\gamma| \leq 1)$.

Now let $\beta = (\beta_1, \dots, \beta_n)$ be a multi-index and $(j_1, \dots, j_{|\beta|})$ denotes the sequence $(\underbrace{1, \dots, 1}_{\beta_1 \text{ times}}, \underbrace{2, \dots, 2}_{\beta_2 \text{ times}}, \dots, \underbrace{n, \dots, n}_{\beta_n \text{ times}})$.

Then by induction on $|\beta|$ we get

$$(1.6) \quad \begin{aligned} \tilde{p}_\beta^i(x, \xi, p) &= \xi^\lambda \left[\frac{\partial^{|\beta|+1} f^i}{\partial x^\lambda \partial x^{\lambda_1} \dots \partial x^{\lambda_{|\beta|}}} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{\lambda_{|\beta|}}}{\partial \tilde{x}^{j_{|\beta|}}} + a_{\beta, \lambda} \right] \\ &\quad + a_{\beta, \lambda}^\gamma p_\gamma^\lambda, |\gamma| \leq |\beta|, \end{aligned}$$

where a are C^∞ functions of $(D^\gamma f: |\gamma| \leq |\beta|)$. Then we claim

- (1) $F^{m-1}(\Delta^{m-1}) = \tilde{\Delta}^{m-1}$ and
- (2) $F_*^{m-1}(\mathcal{D}) = \tilde{\mathcal{D}}$.

Proof of claim.

- (1) A C^1 vector field $X = \xi^i(\partial/\partial x_i)$ is an infinitesimal automorphism of P if and only if $f_* X$ is an infinitesimal automorphism of \tilde{P} . This implies that $F^1(\Delta^1) = \tilde{\Delta}^1$. Then it is clear that $F^k(\Delta^k) = \tilde{\Delta}^k$ for $k = 2, \dots, m-1$.
- (2) For each $k = 1, \dots, m-1$, we have $(F^k)^*(\tilde{\Omega}^k) = \Omega^k$ which is immediate from the definition of the contact system. In particular

$(F^{m-1})^*(\tilde{\Omega}^{m-1}) = \Omega^{m-1}$. Thus we have

$$\begin{aligned} v \in \mathcal{D} &\Leftrightarrow v \in T(\Delta^{m-1}) \quad \text{and } v \text{ annihilates } \Omega^{m-1} \\ &\Leftrightarrow F_*v \in T(\tilde{\Delta}^{m-1}) \quad \text{and } F_*v \text{ annihilates } \tilde{\Omega}^{m-1} \\ &\Leftrightarrow F_*^{m-1}v \in \tilde{\mathcal{D}}, \quad \text{Q.E.D.} \end{aligned}$$

Now we compute $F^*\tilde{\Omega}_\beta^i$, $|\beta| = m - 1$:

$$\begin{aligned} (1.7) (F^{m-1})^*\tilde{\Omega}_\beta^i &= (F^{m-1})^*(d\tilde{p}^i - \tilde{H}_{(\beta,j)}^i(\tilde{x}, \tilde{\xi}, \tilde{p})d\tilde{x}^j) \\ &\quad \text{substitute (1.3)-(1.6) for } \tilde{x}, \tilde{\xi} \text{ and } \tilde{p}, \text{ respectively,} \\ &= \left[\frac{\partial^m f^i}{\partial x^\lambda \partial x^{\lambda_1} \dots \partial x^{\lambda_{m-1}}} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{\lambda_{m-1}}}{\partial \tilde{x}^{j_{m-1}}} + a_{\beta,\lambda} \right] d\xi^\lambda \\ &\quad + a_{\beta,\lambda}^\gamma dp_\gamma^\lambda, \quad |\gamma| \leq m - 1 \\ &\quad + \left[\xi^\lambda \frac{\partial^{m+1} f^i}{\partial x^\lambda \partial x^{\lambda_1} \dots \partial x^{\lambda_{m-1}} \partial x^k} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{\lambda_{m-1}}}{\partial \tilde{x}^{j_{m-1}}} + b_{\beta,k}^i \right] dx^k \end{aligned}$$

where a are C^∞ functions of $(D^\gamma f: |\gamma| \leq m - 1)$ and b are C^∞ functions of $(x, \xi, p, D^\gamma f: |\gamma| \leq m)$.

By the proof of Lemma 1.4, $\tilde{\mathcal{D}}$ on $\tilde{\Delta}^{m-1}$ is given by

$$\begin{cases} \tilde{\Omega}^{m-1} = 0 \\ \tilde{\Omega}_\beta^i = 0, \quad \forall i = 1, \dots, n, \quad \forall \beta \text{ with } |\beta| = m - 1. \end{cases}$$

Recall that $\tilde{\Omega}^{m-1}$ is the contact system and $\tilde{\Omega}_\beta^i$ are the 1 forms defined by the complete system. Since $F_*^{m-1}(\mathcal{D}) = \tilde{\mathcal{D}}$, $(F^{m-1})^*\tilde{\Omega}_\beta^i$ is a linear combination of $\{\omega^i, \omega_\gamma^i, \Omega_\delta^i: i = 1, \dots, n, |\gamma| < m - 1, |\delta| = m - 1\}$, where ω are contact forms. So we set

$$\begin{aligned} (1.8) \quad (F^{m-1})^*\tilde{\Omega}_\beta^i &= c_{\beta,\lambda}^i \omega^\lambda + c_{\beta,\lambda}^{i,\gamma} \omega_\gamma^\lambda + c_{\beta,\lambda}^{i,\delta} \Omega_\delta^\lambda \\ &= c_{\beta,\lambda}^i (d\xi^\lambda - p_k^\lambda dx^k) + c_{\beta,\lambda}^{i,\gamma} (dp_\gamma^\lambda - p_{(\gamma,k)}^\lambda dx^k) \\ &\quad + c_{\beta,\lambda}^{i,\delta} (dp_\delta^\lambda - H_{(\delta,k)}^\lambda dx^k), \end{aligned}$$

where c are C^1 functions on Δ^{m-1} , $|\gamma| \leq m - 2$ and $|\delta| = m - 1$.

By equating the components of $d\xi$ and dp in (1.7) and (1.8) we get c 's as C^∞ functions of $(x, \xi, p, D^\gamma f: |\gamma| \leq m)$ for $(x, \xi, p) \in \Delta^{m-1}$. Substitute this in (1.8) and equate the components of dx^k in (1.7) and (1.8) to get

$$\begin{aligned} (1.9) \quad &\xi^\lambda \frac{\partial^{m+1} f^i}{\partial x^\lambda \partial x^{\lambda_1} \dots \partial x^{\lambda_{m-1}} \partial x^k} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{\lambda_{m-1}}}{\partial \tilde{x}^{j_{m-1}}} \\ &= C^\infty \text{ function of } (x, \xi, p, D^\gamma f: |\gamma| \leq m), \end{aligned}$$

where $(x, \xi, p) \in \Delta^{m-1}$. Since (1.1) is a system of linear partial differential equations of first order obtained from the structure equations of the Lie Algebra \mathcal{G} , we see that $dx^1 \wedge \cdots \wedge dx^n \wedge d\xi^1 \cdots d\xi^n \neq 0$ on $\Delta^1 \subseteq J^1$ and therefore on $\Delta^{m-1} \subseteq J^{m-1}$. Thus there exists a C^∞ function $p(x, \xi)$ such that $(x, \xi, p(x, \xi)) \in \Delta^{m-1}, \forall (x, \xi)$. For each $j = 1$, the restriction of (1.9) to the submanifold $\{(x, \xi, p(x, \xi)): \xi = (0, \dots, 0, \underbrace{1}_{jth}, 0, \dots, 0)\}$ is

$$(1.10) \quad \frac{\partial^{m+1} f^i}{\partial x^j \partial x^{\lambda_1} \dots \partial x^{\lambda_{m-1}} \partial x^k} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{\lambda_{m-1}}}{\partial \tilde{x}^{j_{m-1}}} = C^\infty \text{ function of } (x, D^\gamma f: |\gamma| \leq m).$$

Here $i, j, j_1, \dots, j_{m-1}$ and k are arbitrary. Since the matrix

$$(\partial x^i / \partial \tilde{x}^j)_{i,j=1, \dots, n}$$

is nonsingular and each $\partial x^i / \partial \tilde{x}^j$ is a C^∞ function of (Df) , from (1.10) we get

$$\frac{\partial^{m+1} f^i}{\partial x^j \partial x^{j_1} \dots \partial x^{j_{m-1}} \partial x^k} = C^\infty \text{ function of } (x, D^\gamma f: |\gamma| \leq m).$$

This completes the proof.

2. G-STRUCTURES OF FINITE ORDER

Let G be a Lie subgroup of $Gl(n; \mathbf{R})$ and \mathcal{G} be the associated Lie algebra. The k th prolongation $\mathcal{G}^{(k)}$ of \mathcal{G} is the space of symmetric multilinear mappings

$$t: \underbrace{\mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{(k+1) \text{ times}} \rightarrow \mathbf{R}^n$$

such that, for each fixed $v_1, \dots, v_k \in \mathbf{R}^n$, the linear transformation

$$v \in \mathbf{R}^n \mapsto t(v, v_1, \dots, v_k) \in \mathbf{R}^n \text{ belongs to } \mathcal{G}.$$

G is said to be of finite order k if $\mathcal{G}^{(k)} = 0$ and $\mathcal{G}^{(k-1)} \neq 0$. Riemannian structures and conformal structures (when dimension ≥ 3) are of finite order 1 and 2, respectively (cf. [5 or 6]).

Theorem 2.1. *Let M be a C^∞ manifold of dimension n and P be a G -structure on M . If G is of finite order $m - 1$ ($m \geq 1$), then P is of Frobenius type of order m .*

Proof. Since \mathcal{G} is a linear subspace of $gl(n; \mathbf{R})$, it is defined by $\mathcal{G} = \{(y_j^i) \in gl(n; \mathbf{R}): \sum_{i,j=1}^n c_{i\lambda}^j y_j^i = 0, \lambda = 1, \dots, N\}$, where the $c_{i\lambda}^j$ are constants and N is the codimension of \mathcal{G} in $gl(n; \mathbf{R})$. Therefore, as a linear space, $\mathcal{G}^{(m-1)}$ is isomorphic to the subspace of

$$\mathbf{R}^{n^{m+1}} = (y_{j_1 \dots j_m}^i), \text{ each } i, j \in \{1, \dots, n\},$$

which is defined by the following system of linear equations:

$$(2.11) \quad \sum_{i,j_1=1}^n c_{i\lambda}^{j_1} y_{j_1 \dots j_m}^i = 0, \quad \lambda = 1, \dots, N$$

and

$$\left. \begin{aligned} y_{j_1 j_2 \dots j_m}^i - y_{j_2 j_1 \dots j_m}^i &= 0 \\ \dots & \\ y_{j_1 \dots j_{m-1} j_m}^i - y_{j_1 \dots j_m j_{m-1}}^i &= 0 \end{aligned} \right\} \begin{array}{l} \text{symmetry in} \\ \text{subscripts} \end{array}$$

Since the only solution of (2.11) is $y = 0$, there exists n^{m+1} independent equations in (2.11). Let

$$(2.12) \quad g^1(y) = 0, \dots, g^{n^{m+1}}(y) = 0.$$

Now let M be a C^∞ manifold of dimension n with a G -structure P . We fix a frame (e_1, \dots, e_n) belonging to P . Let $X = \sum_{i=1}^n \xi^i e_i$ be an infinitesimal automorphism of P .

Define ξ_j^i by $[e_j, X] = \sum_{i=1}^n \xi_j^i e_i$. Then the matrix $[\xi_j^i]$ belongs to \mathcal{G} . For any sequence (j_2, \dots, j_k) , each $j \in \{1, \dots, n\}$, we denote by $\xi_{j_2 \dots j_k}^i$ the derivative of $e_{j_k} \dots e_{j_2}(\xi_j^i)$. Then in the Jacobi identity

$$[e_k, [e_j, X]] - [e_j, [e_k, X]] = [[e_k, e_j], X]$$

substitute $\sum_{i=1}^n \xi_j^i e_i$ and $\sum_{i=1}^n \xi_k^i e_i$ for $[e_j, X]$ and $[e_k, X]$, respectively, we get $\xi_{jk}^i - \xi_{kj}^i = \langle \xi^\lambda, \xi^\mu \rangle$, where $\langle \cdot, \cdot \rangle$ denotes a linear combination of the variables inside with C^∞ coefficients. By induction on the number of the subscripts we see that a transposition for any two subscripts in $\xi_{j_1 \dots j_k}^i$ makes a difference by a linear combination of $\{\xi_J^\lambda : |J| < k, \lambda = 1, \dots, n\}$, where $J = (j_1 j_2 \dots)$ is a sequence of subscripts and $|J|$ is the size of J . Moreover, since \mathcal{G} is a linear space, for each fixed $j_2 \dots j_k$, the matrix of the derivatives

$$[\xi_{j_1 j_2 \dots j_k}^i]_{i,j_1=1, \dots, n} \text{ belongs to } \mathcal{G}.$$

Now at each point $x \in M$, consider

$$(2.13) \quad \sum_{i,j_1=1}^n c_{i\lambda}^{j_1} \xi_{j_1 j_2 \dots j_m}^i = 0, \lambda = 1, \dots, N$$

and

$$\left. \begin{aligned} \xi_{j_1 j_2 \dots j_m}^i - \xi_{j_2 j_1 \dots j_m}^i + \delta_{j_1 j_2}^i &= 0 \\ \xi_{j_1 \dots j_{m-1} j_m}^i - \xi_{j_1 \dots j_m j_{m-1}}^i + \delta_{j_{m-1} j_m}^i &= 0 \end{aligned} \right\} *$$

* is the symmetry in the subscripts modulo lower order terms δ , where each δ is a linear combination of $\{\xi_J^t : |J| \leq m-1, t = 1, \dots, n\}$ with C^∞ coefficients. Let

$$(2.14) \quad g^1(x, \xi) = 0, \dots, g^{n^{m+1}}(x, \xi) = 0,$$

be the equations corresponding to (2.12). Since the last n^{m+1} columns of the Jacobian matrix $(\partial g(x, \xi)/\partial \xi)$ is equal to $(\partial g/\partial y)$, which is nonsingular, we can solve (2.14) to get $\xi_{j_1, \dots, j_m}^i =$ a linear combination of $\{\xi_J^t : |J| \leq m-1, t = 1, \dots, n\}$ with C^∞ coefficients, for each i, j_1, \dots, j_m . This completes the proof.

3. REMARKS ON CR STRUCTURES

A CR structure P of complex dimension n and CR codimension d on a C^∞ manifold M is dimension $2n+d$ is a G -structure, where G is the group of all the matrices of the form

$$\begin{bmatrix} A & * \\ 0 & B \end{bmatrix}, \text{ where } A \in GL(n; \mathbf{C}) \subset GL(2n; \mathbf{R}) \text{ and } B \in GL(d; \mathbf{R}).$$

Let (e_1, \dots, e_{2n+d}) be a frame field belonging to P . Let $H(M)$ be the sub-bundle of the tangent bundle $T(M)$ spanned by (e_1, \dots, e_{2n}) . We assume the integrability condition on $H(M)$, as usual (cf. [2]). This group G is of infinite order, for the associated Lie algebra of G contains a matrix or rank 1 (Proposition 1.4 of [5]). However, under certain conditions on the Levi form P is of Frobenius type. When $d = 1$ the following is well known: If M and \widetilde{M} are C^∞ CR manifolds with nondegenerate Levi forms, and $f: M \rightarrow \widetilde{M}$ is a CR diffeomorphism then

- (1) f is locally determined by a finite number of constants, and
- (2) $f \in C^7$ implies that $f \in C^\infty$.

These are consequences of the existence of the invariant Cartan connection on the bundle of pseudoconformal frames over M (cf. [2, 7]). From the viewpoint of this paper the above (1) and (2) follow from the fact that a nondegenerate CR structure of CR codimension 1 is of Frobenius type of order 3. This can be easily shown by the method used in [3] and [4], where direct construction of complete system of mappings have been treated.

When M and \widetilde{M} are real analytic hypersurfaces in \mathbf{C}^n , regularity of CR mappings has been studied in [1] and [7] in relation to the problem of holomorphic extension of CR mappings. For abstract CR manifolds, the greater the CR codimension d is, the more conditions on P are required in order for mappings to be regular. Our further problem is to find conditions for given n and d , under which CR structure P becomes Frobenius type, or more generally, to find conditions on a nonelliptic G -structure (cf. [5]) which imply the regularity of transformations.

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