INVARIANT SUBMANIFOLDS FOR SYSTEMS OF VECTOR FIELDS OF CONSTANT RANK

HEUNGJU AHN[1] AND CHONG-KYU HAN[2]

ABSTRACT. Given a system of vector fields on a smooth manifold that spans a plane field of constant rank, we present a systematic method and an algorithm to find submanifolds that are invariant under the flows of the vector fields. We present examples of partition into invariant submanifolds, which further gives partition into orbits. We use the method of generalized Frobenius theorem by means of exterior differential systems.

1. INTRODUCTION

Let M be a smooth (C^∞) connected manifold of dimension m. We consider an open cover $\{U_\alpha\}_{\alpha\in A}$, for some index set A, and a smooth vector field X_α on U_α, for each $\alpha \in A$. Let G be the pseudogroup of local diffeomorphisms generated by the 1-parameter groups

$$\exp tX_\alpha, \quad -\epsilon < t < \epsilon,$$

whose infinitesimal generator is X_α, where $\epsilon > 0$ depends on α and the domain of a local diffeomorphism. A submanifold $N \subset M$ is said to be invariant if for any $x \in N$ we have $g(x) \in N$ for all $g \in G$.

A point $y \in M$ is said to be reachable from $x \in M$, denoted by $x \sim y$, if $y = g(x)$ for some $g \in G$. $x \sim y$ is an equivalence relation. Its equivalence class S_x, the set of all points reachable from x, is called the orbit of x. An orbit is a submanifold (cf. [Sus, Theorem 4.1]), not necessarily an embedding (cf. [AS, Example 5.5]). An orbit is an invariant submanifold. The orbits of an arbitrary set of vector fields have been studied because of their importance in control theory. If the smallest Lie algebra (a module that is closed under the bracket) that contains all X_α has the maximal rank m, then the orbit of any point is the whole manifold M by Chow's theorem (cf. [Chow], [Kre]). We refer the readers to [Jur], [Sus] and [BMS] for further results on the orbits of vector fields. In this paper we are concerned with the problem of finding invariant submanifolds assuming the following:

Constant rank condition: the linear span of $\{X_\alpha, \alpha \in A\}$ has constant dimension p, for some p, $1 \le p \le m-1$.

This is a strong condition on the topology of M. For instance, the constant rank condition with $p = 1$ holds if and only if M has the Euler characteristic

---

Date: November 27, 2012.

1991 *Mathematics Subject Classification.* Primary 34H05, 37C10, 57R27, 93B05, 93C10.

Key words and phrases. control system, vector fields, reachability, orbits, invariant submanifolds.

First author was partially supported by National Research Foundation (NRF) of Republic of Korea, with grant number 2010-0024633, and Second author with grant number 2011-0008976.

zero (cf. [MS]). Once the constant rank condition is assumed to find the invariant submanifolds is a local problem. Our object of study is indeed a sub-bundle T' of rank p of the tangent bundle TM spanned by X_α. Suppose that

$$\{X_1, \dots, X_p\} \tag{1.1}$$

spans T' locally. If

$$[T', T'] \subset T' \tag{1.2}$$

holds, which means that the module generated by (1.1) is closed under the bracket, by applying the Frobenius theorem locally, we see that M is foliated by orbits of dimension p. The minimal dimension of an orbit is p. If (1.2) does not hold we may reduce the control system given by (1.1) to invariant submanifolds. By partitioning M into a disjoint union of invariant submanifolds one may obtain a finer partition by orbits as shown in the examples in §4.

Suppose that N is locally defined as a common zero locus of a non-degenerate set (definition in §2) of smooth real-valued functions $\rho = (\rho^1, \dots, \rho^d)$, $d \geq 1$, where d is the codimension of N. A local characterization for N to be invariant is that

$$(X_j \rho)(x) = 0, \quad j = 1, \dots, p, \quad \text{for all } x \text{ with } \rho(x) = 0. \tag{1.3}$$

If $p \geq 2$, then (1.3) is over-determined, hence there are no such ρ generically. In the case $p = 1$ the classical first integrals $\rho = (\rho^1, \dots, \rho^{m-1})$ satisfies $X_j \rho = 0$. Thus every level set of a non-degenerate first integral is an invariant submanifold. Furthermore, every common zero locus of a non-degenerate set of first integrals is an invariant submanifold. In solving (1.3) we use the method of exterior differential systems (cf. §2) and the generalizations of the Frobenius theorem as in §3.

2. Preliminaries

In this section, we review some of the basic facts of the exterior differential system and the theory of the first integrals due to E. Cartan and R. Gardner. Then we define the generalized first integral, or functions with invariant zero locus, which has been used in [AH, HL, HP1, HP2]. In this section our argument is purely local, thus M is a small open set of a smooth manifold of dimension m, which is allowed to shrink finitely many times as our argument proceeds. We fix notations first: let

$$\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^m ,$$

be the exterior algebra of smooth differential forms, where Ω^0 is the ring of smooth real-valued functions and Ω^r, $r = 1, \dots, m$, is the module over Ω^0 of smooth r-forms on M.

Definition 2.1. A subalgebra $\mathscr{I}$ is called an ideal (algebraic ideal) if the following conditions hold:

(i) $\mathscr{I} \wedge \Omega^* \subset \mathscr{I}$.
(ii) Homogeneity condition: If $\phi := \sum_{r=0}^m \phi_r \in \mathscr{I}$, where $\phi_r \in \Omega^r$, then each component $\phi_r \in \mathscr{I}$.

The homogeneity condition implies that $\mathscr{I}$ is a two-sided ideal, that is,

$$\Omega^* \wedge \mathscr{I} \subset \mathscr{I}$$

.

A system of C^∞ real-valued functions $r^1,\dots,r^d$ are said to be *non-degenerate* if $dr^1\wedge\cdots\wedge dr^d\neq 0$. Let $\theta=(\theta^1,\dots,\theta^s)$ be a system of smooth 1-forms. Rank of θ at $x\in M$ is the dimension of the linear span $\langle\theta^1,\dots,\theta^s\rangle$. If θ has constant rank on a submanifold $N\subset M$, then $\langle\theta\rangle^\perp$, the set of vectors tangent to M that are annihilated by θ, forms a vector bundle over N, where the fibre at $x\in N$ is a subspace of the tangent space of M at x.

Definition 2.2. An ideal $\mathscr{I}$ is said to be regular if $\mathscr{I}$ is generated by a non-degenerate system of real-valued functions $r:=(r^1,\dots,r^d)$ and a finite set of smooth 1-forms $\theta:=(\theta^1,\dots,\theta^s)$ of constant rank. A regular ideal generated by r and θ shall be denoted by $\mathscr{I}(r,\theta)$ or by $\{r,\theta\}$. For $\phi,\psi\in\Omega^*$ we write

$$\phi\equiv\psi \mod \{r,\theta\}$$

if and only if $\phi-\psi\in\{r,\theta\}$.

A regular ideal $\{r,\theta\}$ corresponds in one-to-one manner to a vector bundle $\mathscr{V}\to N$, where N is the common zero set of r, and $\mathscr{V}=\langle\theta\rangle^\perp$ as in the following diagram:

$$\begin{array}{ccc} \mathscr{V} & \hookrightarrow & TM \\ \downarrow & & \downarrow \\ N & \hookrightarrow & M \end{array}$$

Suppose that the regular ideals $\{r,\theta\}$ and $\{\tilde r,\tilde\theta\}$ define vector bundles $\mathscr{V}\to N$ and $\tilde{\mathscr{V}}\to\tilde N$, respectively. We define the inclusion $(\mathscr{V}\to N)\subset(\tilde{\mathscr{V}}\to\tilde N)$ by

$$N\subset\tilde N \quad\text{and}\quad \mathscr{V}_x\subset\tilde{\mathscr{V}}_x,\quad \forall x\in N.$$

We observe that $(\mathscr{V}\to N)\subset(\tilde{\mathscr{V}}\to\tilde N)$ if and only if

$$\{r,\theta\}\supset\{\tilde r,\tilde\theta\}.$$

Remark 2.3. Let $\{r,\theta\}\subset\Omega^*$ be a regular ideal and ω be a k-form for some $k=1,\dots,m$. Then $\omega\in\{r,\theta\}$ if and only if $\omega(V_1,\dots,V_k)=0$, for any $V_1,\dots,V_k$ in the same fibre $\mathscr{V}_x$, $x\in N$. If N is the common zero locus of a non-degenerate $r=(r^1,\dots,r^d)$, the tangent bundle TN corresponds to the ideal $\{r,dr\}$.

We consider a set of smooth vector fields

$$X_1,\dots,X_p, \tag{2.4}$$

that are linearly independent at all points of M. A smooth real-valued function ρ is called a *first integral* of (2.4) if ρ remains constant along integral curves of each X_j, $j=1.\dots,p$, that is, if

$$X_j\rho=0,\quad j=1,\dots,p. \tag{2.5}$$

There can be at most $s:=m-p$ first integrals $\rho^1,\dots,\rho^s$ with $d\rho^1\wedge\cdots\wedge d\rho^s\neq 0$, which is the case that the Frobenius integrability condition (1.2) holds with $T'=<X_1,\dots,X_p>$. Now let

$$\theta=(\theta^1,\dots,\theta^s),\quad\text{where}\quad s=m-p, \tag{2.6}$$

be a system of independent smooth 1-forms that annihilate X_j, $j=1,\cdots,p$. Then (1.2) is equivalent to

$$d\theta\equiv 0,\quad \mod \{\theta\}. \tag{2.7}$$

By a Pfaffian system we shall simply mean a system of independent 1-forms. Pfaffian system (2.6) is said to be integrable, or involutive, if (2.7) holds. It is easy to see the following

Proposition 2.4. *Let $X_1,\dots,X_p$ be smooth independent vector fields and $\theta = (\theta^1,\dots,\theta^s)$, $s = m-p$, be a system of smooth independent* 1*-forms as in* (2.4) *and* (2.6). *Then for a smooth real-valued function ρ with $d\rho \neq 0$ the following are equivalent:*

(i) *ρ is a first integral;*
(ii) *X_j, $j = 1,\dots,p$, is tangent to the level sets of ρ;*
(iii) *ρ satisfies*

$$d\rho \in \{\theta\}. \tag{2.8}$$

Next we review the notion of the derived flag of Pfaffian systems as in [Gar]. Given a system $\theta = (\theta^1,\dots,\theta^s)$ of independent 1-forms, we denote by I the sub-bundle of $T^*(M)$ spanned by θ. We regard I also as a submodule of Ω^1 generated by θ. Let $d : I \to \Omega^*$ be the exterior differentiation and $\pi : \Omega^* \to \Omega^*/I$ be the projection. Then

$$\delta = \pi \circ d : I \to \Omega^2/I \tag{2.9}$$

is a module homomorphism. Let $I^{(1)} \subset I$ be the kernel of δ. Then we have the following exact sequence

$$0 \longrightarrow I^{(1)} \longrightarrow I \xrightarrow{\delta} dI/I \longrightarrow 0.$$

$I^{(1)}$ is called the *first derived system*. Inductively, assuming that $I^{(r-1)}$ has constant rank, we construct the r-th derived system by the exactness of

$$0 \longrightarrow I^{(r)} \longrightarrow I^{(r-1)} \xrightarrow{\delta} dI^{(r-1)}/I^{(r-1)} \longrightarrow 0.$$

Again we assume that $I^{(r)}$ has constant rank, that is, a sub-bundle of $T^*(M)$. Then there will be a smallest integer ν such that $I^{(\nu+1)} = I^{(\nu)}$. We call

$$I = I^{(0)} \supset I^{(1)} \supset I^{(2)} \supset \cdots \supset I^{(\nu-1)} \supset I^{(\nu)}$$

the *derived flag* of I and ν the *derived length*. Note that $I^{(\nu)}$ is the largest involutive subsystem of I.

A basic observation in [Gar] is that a function ρ is a first integral if and only if $d\rho \in I^{(\nu)}$. Let q be the rank of $I^{(\nu)}$. Then by the Frobenius theorem there exist q first integrals that are functionally independent. If the rank of $I^{(\nu)}$ is non-zero the Pfaffian system θ is said to be of type ν. If $I^{(\nu)}$ has rank zero, the Pfaffian system θ is said to be of infinite type. The condition that θ has type ν is given as a system of non-linear partial differential equations of order $\nu+1$ for the coefficients of θ (cf. [HK]).

Now we define the *generalized first integral* to be a function ρ that satisfies (2.5) on its zero set, namely,

Definition 2.5. Let $\rho = (\rho^1,\dots,\rho^d)$, $d \le s$, be a non-degenerate set of smooth real-valued functions. We say that ρ is a system of *generalized first integrals* if

$$d\rho^\delta \in \{\rho,\theta\}, \quad \text{for each } \delta = 1,\dots,d. \tag{2.10}$$

Compare (2.10) with the notion of first integral (2.8). We also observe that (2.10) is equivalent to one of the following:

(i) $(X_j\rho^\delta)(x) = 0$, $\quad \delta = 1,\dots,d$, for all $x \in M$ with $\rho(x) = 0$;
(ii) X_j are tangent to the common zero set of ρ.

Our standard reference of the theory of exterior differential system is [BCG3].

3. Invariant Submanifolds

In this section, we discuss the existence of invariant submanifolds for (2.4) that spans a rank p sub-bundle T' of TM. Let $\theta = (\theta^1,\dots,\theta^s)$ be 1-forms as in (2.6). The torsion tensor for θ is by definition $d\theta^\alpha$ mod $\{\theta\}$. Invariant submanifolds are the zero sets of generalized first integrals. Our strategy is to find a function or a system of functions ρ that can possibly be a generalized first integral by linear algebraic investigation of the torsion tensor for (2.6) and then to check whether thus found ρ indeed satisfies (2.8) or (2.10). To represent the torsion tensor in matrices let $\omega^1,\dots,\omega^p$ be a set of closed 1-forms that completes θ to a local coframe

$$(\theta^1,\dots,\theta^s,\omega^1,\dots,\omega^p). \tag{3.11}$$

Set

$$d\theta^\alpha \equiv \sum_{1\le i<j\le p} T^\alpha_{ij}\omega^i\wedge\omega^j, \quad \text{mod } \{\theta\}, \quad \alpha = 1,\dots,s.$$

Arranging the pairs (i,j) with $i<j$ in lexicographical order, we write in matrices as

$$\begin{bmatrix} d\theta^1 \\ \vdots \\ d\theta^s \end{bmatrix} \equiv \underbrace{\begin{bmatrix} T^1_{12} & T^1_{13} & \cdots & T^1_{p-1,p} \\ \vdots & & & \vdots \\ T^s_{12} & T^s_{13} & \cdots & T^s_{p-1,p} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} \omega^1\wedge\omega^2 \\ \omega^1\wedge\omega^3 \\ \vdots \\ \omega^{p-1}\wedge\omega^p \end{bmatrix}, \quad \text{mod } \{\theta\}. \tag{3.12}$$

The matrix $\mathcal{T}$ of size $s\times\binom{p}{2}$ shall be called the *torsion matrix* with respect to the coframe (3.11). $\mathcal{T}=0$ is the Frobenius integrability condition, which is the case of type 0. In general, we construct the derived flag as follows: Suppose that $\phi = \sum_{\alpha=1}^s a_\alpha\theta^\alpha$ is in the kernel of δ as in (2.9), that is, $d\phi\equiv 0$, mod $\{\theta\}$. Since

$$d\phi \equiv \sum_{\alpha=1}^s a_\alpha d\theta^\alpha, \quad \text{mod } \{\theta\},$$

by substituting (3.12) for $d\theta^\alpha$ we have

$$(a_1,\dots,a_s)\underbrace{\begin{bmatrix} T^1_{12} & T^1_{13} & \cdots & T^1_{p-1,p} \\ \vdots & & & \vdots \\ T^s_{12} & T^s_{13} & \cdots & T^s_{p-1,p} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} \omega^1\wedge\omega^2 \\ \omega^1\wedge\omega^3 \\ \vdots \\ \omega^{p-1}\wedge\omega^p \end{bmatrix} \equiv 0, \quad \text{mod } \{\theta\}. \tag{3.13}$$

Since the 2-forms $\omega^j\wedge\omega^k$, $j<k$, are independent (3.13) implies that the row vector $(a_1,\dots,a_s)$ is in the kernel of the multiplication by $\mathcal{T}$ on the right. Thus by finding the basis of the kernel we obtain the generators of $I^{(1)}$. Inductively, if $\phi = (\phi^1,\dots)$ are the generators of $I^{(k)}$ then $I^{(k+1)}$ is found by finding the null vectors of the torsion tensor $d\phi$ mod ϕ.

Now we discuss how to find the generalized first integrals: Suppose that $\rho = (\rho^1,\dots,\rho^d)$ is a generalized first integral. Then (2.10) implies

$$d\rho^\delta = \sum_{\alpha=1}^s a^\delta_\alpha\theta^\alpha, \qquad \text{mod } \{\rho\}, \quad \delta = 1,\dots,d, \tag{3.14}$$

for some functions a^δ_α. Applying d to (3.14), we obtain

$$0 \equiv \sum_\alpha a^\delta_\alpha d\theta^\alpha, \qquad \mathrm{mod}\ \{\rho,\theta\} \tag{3.15}$$

Substitute (3.12) for $d\theta^\alpha$ in (3.15), to obtain

$$0 \equiv \sum_{\alpha=1}^{s} a^\delta_\alpha \sum_{1\le i<j\le p} T^\alpha_{ij}\omega^i\wedge\omega^j, \quad \mathrm{mod}\ \{\rho,\theta\}, \quad \delta = 1,\dots,d,$$

which can be written in matrices as

$$\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} \equiv \underbrace{\begin{bmatrix}a^1_1 & \cdots & a^1_s\\ \vdots & & \vdots\\ a^d_1 & \cdots & a^d_s\end{bmatrix}}_{\mathscr{A}} \underbrace{\begin{bmatrix}T^1_{12} & T^1_{13} & \cdots & T^1_{p-1,p}\\ \vdots & & & \vdots\\ T^s_{12} & T^s_{13} & \cdots & T^s_{p-1,p}\end{bmatrix}}_{\mathscr{T}} \begin{bmatrix}\omega^1\wedge\omega^2\\ \omega^1\wedge\omega^3\\ \vdots\\ \omega^{p-1}\wedge\omega^p\end{bmatrix}, \ \mathrm{mod}\ \{\rho,\theta\}. \tag{3.16}$$

Since $d\rho^1\wedge\cdots\wedge d\rho^d \neq 0$, $\mathscr{A}$ has maximal rank d and $\omega^i\wedge\omega^j$ with $i<j$ are independent 2-forms (3.16) implies $\mathscr{A}\mathscr{T}=0$. Hence $\mathscr{T}$ has rank at most $(s-d)$ on the zero set of ρ. Thus a necessary condition for $\rho=(\rho^1,\dots,\rho^d)$ to be a generalized first integral is that the null space of the torsion matrix on the zero locus of ρ has dimension d, that is, rank of $\mathscr{T}$ is at most $s-d$ on the zero locus of ρ. In the cases that $s \le \binom{p}{2}$, this rank condition is equivalent to that the square submatrices of size $s-d+1$ have determinants zero. By factoring those determinants we find ρ such that all those determinants are elements of the ideal $\{\rho\}$.

Summarizing our observations we find the first integrals and the generalized first integrals as follows:

i) Find null vectors of $\mathscr{T}$, that is, vectors $A=(a_1,\dots,a_s)$ such that $A\mathscr{T}=0$. Then 1-forms $\phi=\sum_{\alpha=1}^s a_\alpha\theta^\alpha$ span the first derived system $I^{(1)}$. Then the Frobenius integrability

$$[I^{(1)},I^{(1)}]\subset I^{(1)}$$

is equivalent to

$$d\phi^\delta = 0, \quad \mathrm{mod}\ \{\phi\}, \tag{3.17}$$

where $\phi=(\phi^1,\dots,\phi^d)$, d is the rank of $I^{(1)}$ and $\delta=1,\dots,d$. If (3.17) holds then d is the number of independent first integrals and the Pfaffian system θ is of type 1. If (3.17) does not hold, repeat the same process starting with $I^{(1)}$, to construct $I^{(2)}$ and inductively the derived flag. Then find functions ρ with $d\rho\in I^{(\nu)}$, which are the first integrals.

ii) For a given d with $1\le d\le s$, check the determinants of square submatrices of $\mathscr{T}$ of size $s-d+1$. If there exists a non-degenerate set of real-valued functions $\rho=(\rho^1,\dots,\rho^d)$ so that each of those determinants is in the ideal $\{\rho\}$, then ρ can possibly be a generalized first integral. It is indeed a generalized first integral if ρ satisfies (2.10). Recalling that ρ is obtained from $\mathscr{T}$, which comes from $d\theta$, we see that (2.10) is a second order system of non-linear partial differential equations for θ.

4. Examples

Example 4.1. On $\mathbb{R}^4 = \{(x,y,z,u)\}$ let

$$X = \frac{\partial}{\partial x} + e^u \frac{\partial}{\partial u}, \quad Y = \frac{\partial}{\partial y} + u\frac{\partial}{\partial z}.$$

Then the system of 1-forms

$$\theta^1 = -u\,dy + dz$$
$$\theta^2 = -e^u dx + du$$

annihilates X and Y. We have

$$\begin{pmatrix} d\theta^1 \\ d\theta^2 \end{pmatrix} = \underbrace{\begin{pmatrix} -e^u \\ 0 \end{pmatrix}}_{\mathcal{T}} dx \wedge dy, \quad \text{mod } \{\theta\}$$

and

$$[X,Y] = e^u \frac{\partial}{\partial z}.$$

Since a null vector (a_1, a_2) of $\mathcal{T}$ as in (3.13) is $(0,1)$,

$$\phi = \sum_{\alpha=1}^{2} a_\alpha \theta^\alpha = \theta^2$$

generates $I^{(1)}$. Thus $I^{(1)}$ has rank 1. Since

$$\begin{aligned} d\phi &= -e^u du \wedge dx \\ &\equiv 0, \quad \text{mod } \{\phi\}, \end{aligned}$$

$I^{(1)}$ is involutive, thus the Pfaffian system (θ^1, θ^2) has type 1. In fact $\rho(x,y,z,u) = x + e^{-u}$ is a first integral because

$$d\rho = -e^{-u}\phi \in I^{(1)}.$$

Therefore, $\mathbb{R}^4$ is foliated by the invariant hypersurfaces

$$x + e^{-u} = \text{constant}. \tag{4.18}$$

Since X, Y and $[X,Y]$ are independent each leaf in (4.18) is an orbit by Chow's theorem.

Example 4.2. On $\mathbb{R}^4 = \{(x,y,z,u)\}$ let

$$X = \frac{\partial}{\partial x} + e^u \frac{\partial}{\partial u}, \quad Y = \frac{\partial}{\partial y} + u\frac{\partial}{\partial z} + \frac{\partial}{\partial u}.$$

Then the system of 1-forms

$$\theta^1 = -u\,dy + dz$$
$$\theta^2 = -e^u dx - dy + du$$

annihilates X and Y. By the same arguments as in Example 4.1 we see that the torsion matrix with respect to the coframe $(\theta^1, \theta^2, dx, dy)$ is $\mathcal{T} = \begin{pmatrix} -e^u \\ e^u \end{pmatrix}$, that $\phi = \theta^1 + \theta^2$ generates $I^{(1)}$ and that

$$d\phi = \underbrace{\begin{pmatrix} 0, & e^u, & 1 \end{pmatrix}}_{\mathcal{T}'} \begin{bmatrix} dx \wedge dy \\ dx \wedge du \\ dy \wedge du \end{bmatrix}, \quad \text{mod } \{\phi\}.$$

Hence $I^{(1)}$ is not involutive and $I^{(2)} = \{0\}$, so that the Pfaffian system (θ^1, θ^2) has infinite type. Therefore, there are no first integrals other than constants.

Example 4.3. On $M = \mathbb{R}^3$ consider vector fields $X = \dfrac{\partial}{\partial x}$ and $Y = \dfrac{\partial}{\partial y} + xz\dfrac{\partial}{\partial z}$. Then $\theta = dz - xzdy$ annihilates X and Y. We have

$$\begin{aligned} d\theta &\equiv -zdx \wedge dy - xdz \wedge dy \\ &\equiv -zdx \wedge dy \quad \text{mod } \{\theta\} \\ &\equiv \mathcal{T}\, dx \wedge dy \quad \text{mod } \{\theta\}, \quad \text{where } \mathcal{T}(x,y,z) = -z \end{aligned}$$

Thus $\rho(x,y,z) = z$ is a candidate for a generalized first integral. Now we check (2.10), namely, whether ρ satisfies $0 \neq d\rho \in (\rho, \theta)$: Indeed we have

$$\begin{aligned} d\rho &= dz \\ &\equiv \theta, \quad \text{mod } \{\rho\}. \end{aligned}$$

Therefore, the invariants manifolds are

$$\mathbb{R}^3 = \mathcal{O}_+ \cup \mathcal{O}_0 \cup \mathcal{O}_-, \tag{4.19}$$

where $\mathcal{O}_+ = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$, etc. Since

$$[X,Y] = z\frac{\partial}{\partial z}$$

by Chow's theorem each of $\mathcal{O}_+$, $\mathcal{O}_0$ and $\mathcal{O}_-$ is an orbit. Thus (4.19) is partition of $\mathbb{R}^3$ by orbits.

References

[AH] H. Ahn and C.-K. Han, *Local geometry of Levi-forms associated with the existence of complex submanifolds and the minimality of generic CR manifolds*, J. Geom. Anal., **22** (2012), 561-582.

[AS] A. Agrachev and Y. Sachkov, *Control theory from the geometric viewpoint*, Encyclopaedia of Mathematical Sciences, vol. 87, Springer-Verlag, Berlin, 2004, Control Theory and Optimization, II.

[BCG3] R. Bryant, S.-S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York, 1991.

[BMS] R. Brockett, R. Millman, and H. Sussmann (eds.), *Differential geometric control theory*, Progress in Mathematics, vol. 27, Mass., Birkhäuser Boston, 1983.

[Chow] W.L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann., **117** (1939), 98–105.

[Gar] R. Gardner, *Invariants of Pfaffian systems*, Trans. Amer. Math. Soc. **126** (1967), 514–533.

[HK] C.-K. Han and Hyeseon Kim, *Partial integrability of almost complex structures and some algebraic generalizations of the Newlander-Nirenberg theorem,* J. Korean Math. Soc. **49** (2012), 379-394

[HL] C.-K. Han and K.-H. Lee, *Integrable submanioflds in almost complex manifolds*, J Geom. Anl. **20** (2010), 177-192

[HP1] C.-K. Han and Jongdo Park, *Partial integrability of almost complex structures and the local solvability of quasi-linear Cauchy-Riemann equations*, Pacific J. Math., to appear

[HP2] C.-K. Han and Jongdo Park, *Generalized first integrals and overdetermined systems of quasi-linear partial differential equations of first order*, preprint

[Jur] V. Jurdjevic, *Geometric control theory*, Cambridge Studies in Advanced Mathematics, vol. 52, Cambridge University Press, Cambridge, 1997.

[Kre] A. Krener, *A generalization of Chow's theorem and the bang-bang theorem to non-linear control problems*, SIAM J. Control **12** (1974), 43–52.

[MS] J. Milnor and D. Stasheff, *Characteristic classes*, Ann. of Math. Stud., 76, Princeton Univ. Press, Princeton, N.J., 1974

[Sus] H. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188.

[1] DEPARTMENT OF MATHEMATICS, POSTECH, POHANG 790-784, KOREA
E-mail address: heungju@gmail.com

[2] DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA
E-mail address: ckhan@snu.ac.kr