PARTIAL INTEGRABILITY OF ALMOST COMPLEX STRUCTURES AND THE EXISTENCE OF SOLUTIONS FOR QUASI-LINEAR CAUCHY-RIEMANN EQUATIONS

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Abstract. We study the local solvability of the system of the quasi-linear Cauchy-Riemann equations for d unknown functions in n complex variables, which is a system of elliptic type and overdetermined if $n \geq 2$. We consider an associate almost complex structure on $\mathbb{C}^{n+d}$ and its partial integrability and prove by using the Newlander-Nirenberg theorem and its algebraic generalizations that the existence of a pseudo-holomorphic function on the zero set is equivalent to the local solvability of the original quasi-linear system. We discuss an algorithm of finding pseudo-holomorphic functions on the zero set and then present examples.

Introduction

A classical method for solving partial differential equations (PDE) of first order is the *method of characteristics* which originated from the work of G. Monge (cf. [Mon]). One finds curves along which a PDE becomes a system of ordinary differential equations and construct a solution whose graph, or 1-jet graph, is a union of those curves. Consider a quasi-linear equation

$$\sum_{\lambda=1}^{n} a^{\lambda}(x,u)\frac{\partial u}{\partial x^{\lambda}} = b(x,u) \tag{0.1}$$

for a real-valued function u in n real variables $x = (x^1, \ldots, x^n)$, where a^{λ} and b are smooth (C^{∞}) and $a^{\lambda}, \lambda = 1, \ldots, n$ are not all zero. The

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characteristic vector field of (0.1) is a smooth vector field

$$X = \sum_{\lambda=1}^{n} a^\lambda \frac{\partial}{\partial x^\lambda} + b\frac{\partial}{\partial u}$$

on $\mathbb{R}^{n+1} = \{(x,u)\}$ and a smooth real-valued function $\phi(x,u)$ is a first integral of X if $X\phi = 0$. Then by the implicit function theorem any first integral ϕ with $\phi_u \neq 0$ gives an implicit solution $\phi(x,u) = 0$ to (0.1). The same method works for systems. Consider

$$\sum_{\lambda=1}^{n} a_j^\lambda(x,u)\frac{\partial u}{\partial x^\lambda} = b_j(x,u), \quad j = 1,\dots,p,\ p \le n, \tag{0.2}$$

for a real-valued function u in n real variables $x = (x^1,\dots,x^n)$. We assume the matrix $\left(a_j^\lambda\right)$ has maximal rank p. If $p \ge 2$, then (0.2) is overdetermined, therefore, there are no solutions generically. To discuss the existence of solutions let

$$X_j = \sum_{\lambda=1}^{n} a_j^\lambda \frac{\partial}{\partial x^\lambda} + b_j\frac{\partial}{\partial u}, \quad j = 1,\dots,p, \tag{0.3}$$

be vector fields on $\mathbb{R}^{n+1} = \{(x,u)\}$. For a smooth function $u(x)$ a normal vector to the graph $S = \{(x,u(x)) \in \mathbb{R}^{n+1}\}$ is $(\nabla u, -1) = (\frac{\partial u}{\partial x^1},\dots,\frac{\partial u}{\partial x^n},-1)$. Then (0.2) is equivalent to $X_j \cdot (\nabla u, -1) = 0$, which implies that X_j is tangent to the graph S at every point. A smooth real-valued function F is said to have invariant zero-level with respect to vector fields $X_1,\dots,X_p$ if $(X_jF)(x) = 0$ for all $j = 1,\dots,p$ and for all x with $F(x) = 0$. We have

Theorem 0.1. *Let $(x_0,u_0) \in \mathbb{R}^n \times \mathbb{R}$. On a neighborhood of (x_0,u_0) there exists a solution $u(x)$ of (0.2) with $u(x_0) = u_0$ if and only if there is a function $F(x,u)$ with $\frac{\partial F}{\partial u} \neq 0$ and $F(x_0,u_0) = 0$ that has invariant zero-level with respect to the set of vector fields (0.3).*

Proof. Suppose that $u = f(x)$, $x = (x^1,\dots,x^n)$, is a solution of (0.2). Let $F(x,u) := f(x) - u$. Then we see that $F_u \neq 0$ and that for each $j = 1,\dots,p$

$$X_jF = 0 \quad \text{on} \quad \{F = 0\}.$$

Conversely, suppose that $F(x,u)$ is a function with $F_u \neq 0$, $F(x_0,u_0) = 0$, that has invariant zero-level. Differentiating the implicit function $F(x,u) = 0$ with respect to x^λ using the chain rule, we have

$$\frac{\partial F}{\partial x^\lambda} = -\frac{\partial F}{\partial u}\frac{\partial u}{\partial x^\lambda}, \quad \text{for each } \lambda = 1,\dots,n. \tag{0.4}$$

On the other hand, since F has invariant zero-level we have

$$\begin{aligned} X_j F &= \sum_{\lambda=1}^{n} a_j^\lambda \frac{\partial F}{\partial x^\lambda} + b_j \frac{\partial F}{\partial u} \\ &= 0 \quad \text{on} \quad \{F=0\}. \end{aligned} \tag{0.5}$$

Substituting (0.4) for $\frac{\partial F}{\partial x^\lambda}$ in (0.5) we have

$$-\frac{\partial F}{\partial u}\left(\sum_{\lambda=1}^{n} a_j^\lambda \frac{\partial u}{\partial x^\lambda} - b_j\right) = 0 \quad \text{on} \quad \{F=0\},$$

that is, $u = f(x)$ satisfies (0.2).

□

For more details on (0.2) we refer the readers to [HP]. In this paper we study the complex analogue of Theorem 0.1. Let $z = (z^1, \dots, z^n)$ for $z^j = x^j + \sqrt{-1}y^j$ be complex variables and let

$$\frac{\partial}{\partial \bar{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j}\right) \quad \text{and} \quad \frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}\right).$$

Consider a system of PDE for a complex-valued unknown function $w = w(z, \bar{z})$

$$\frac{\partial w}{\partial \bar{z}^j} + \sum_{k=1}^{n} A_j^k(z, \bar{z}, w, \bar{w}) \frac{\partial w}{\partial z^k} = B_j(z, \bar{z}, w, \bar{w}), \quad j = 1, \dots, n, \tag{0.6}$$

where A_j^k and B_j are complex-valued C^∞ functions that are defined on a neighborhood of the origin of $\mathbb{C}^{n+1} = \{(z, w)\}$ and A_j^k are sufficiently small. If $n \geq 2$, then (0.6) is overdetermined. We shall call (0.6) *quasi-linear Cauchy-Riemann equations.* We prove the local solvability in C^∞ category by purely formal arguments based on the Newlander-Nirenberg theorem and its algebraic generalizations. We observed that a function $\zeta(z, \bar{z}, w, \bar{w}) = 0$ is an implicit solution to (0.6) if and only if ζ is pseudo-holomorphic on the zero set (cf. Definition 1.6) with respect to an almost complex structure J on $\mathbb{C}^{n+1} = \{(z, w)\}$ determined by the coefficients A_j^k and B_j (Theorem 3.3). Another observation is that a function $\zeta(z, \bar{z}, w, \bar{w})$ with $d\zeta \wedge d\bar{\zeta} \neq 0$ is pseudo-holomorphic on the zero set if and only if the zero locus $\zeta = 0$ is a J-invariant submanifold of $(\mathbb{C}^{n+1}, J)$ (Theorem 2.5). To check the partial integrability of the almost complex structure we make use of Theorem 1.3, which is due to L. Nirenberg and F. Treves.

§4 is a generalization of our results of §3 to the cases of multiple unknown functions $w = (w^1, \dots, w^d)$:

$$\frac{\partial w^\alpha}{\partial \bar{z}^j} + \sum_{k=1}^{n} A_j^k(z, \bar{z}, w, \bar{w}) \frac{\partial w^\alpha}{\partial z^k} = B_j^\alpha(z, \bar{z}, w, \bar{w}), \tag{0.7}$$

for each $j = 1, \dots, n$ $(n \geq 1)$ and $\alpha = 1, \dots, d$, where A_j^k and B_j^α are C^∞ functions defined on a neighborhood of the origin of $\mathbb{C}^{n+d}$ and A_j^k are sufficiently small.

We discuss in §5 the determined case $n = 1$. In §6 we present examples of $n = 2$ including the equations for the pseudo-analytic functions, which will be introduced in §7. In the last section of this paper we briefly survey the history of the perturbed Cauchy-Riemann equations and overdetermined PDE systems.

Finally we mention the regularity of solutions to (0.6) or (0.7): It is well-known (cf. [GT]) that a linear elliptic partial differential equation is hypoelliptic, that is, any distribution solution is C^∞ whenever all the coefficients of the differential operators are C^∞. In general, a nonlinear differential equation is said to be elliptic if its linearization is an elliptic differential operator (cf. [Ta]). It was proved in [DN] that a non-linear elliptic systems are hypoelliptic by Schauder type a priori estimates. Since the linearization of each quasi-linear equation in (0.6) or (0.7) is elliptic, any $C^{1,\alpha}$ solution to (0.6) or (0.7) is always C^∞.

1. J-holomorphic functions on almost complex manifolds

Let M be a smooth (C^∞) manifold of dimension $2m$, $m \geq 2$, with smooth almost complex structure J. Then the complexified tangent bundle $\mathbb{C}TM$ has decomposition

$$\mathbb{C}TM = T^{1,0}(M) \oplus T^{0,1}(M),$$

where $T^{1,0}(M)$ ($T^{0,1}(M)$, respectively) is the sub-bundle of rank m of eigen-vectors of J associated with the eigen-value i ($-i$, respectively). The dual decomposition of the complexified cotangent bundle $\mathbb{C}T^*M$ is

$$\mathbb{C}T^*M = (T^*M)^{1,0} \oplus (T^*M)^{0,1}.$$

We can find real vector fields X_j, $j = 1, \dots, m$, such that

$$X_1, JX_1, \dots, X_m, JX_m$$

spans the real tangent bundle TM. Let $Z_j = \frac{1}{2}(X_j - iJX_j)$ and $\bar{Z}_j = \frac{1}{2}(X_j + iJX_j)$, for each $j = 1, \dots, m$. Then $\{Z_1, \dots, Z_m\}$ spans $T^{1,0}(M)$ and $\{\bar{Z}_1, \dots, \bar{Z}_m\}$ spans $T^{0,1}(M)$. Let $\{\theta^1, \dots, \theta^m\}$ be a set of independent 1-forms that annihilates $T^{0,1}(M)$ and thus $\{\bar{\theta}^1, \dots, \bar{\theta}^m\}$ annihilates $T^{1,0}(M)$. Then the sub-bundles $(T^*M)^{1,0}$ and $(T^*M)^{0,1}$ of the complexified cotangent bundle are the linear spans of $\{\theta^1, \dots, \theta^m\}$ and $\{\bar{\theta}^1, \dots, \bar{\theta}^m\}$, respectively.

A complex-valued function ζ is said to be *J-holomorphic* (or *pseudo-holomorphic*) if

$$\bar{Z}_j \zeta = 0, \quad j = 1, \dots, m. \tag{1.1}$$

(1.1) is an overdetermined system of linear PDEs, and thus in general, there are no solutions other than constants. J-holomorphic functions $\zeta^1, \dots, \zeta^q$ are said to be *independent* if

$$d\zeta^1 \wedge \cdots \wedge d\zeta^q \neq 0.$$

(1.1) is equivalent to saying that $d\zeta$ is a section of $(T^*M)^{1,0}$, so that there exist at most m independent J-holomorphic functions. J is said to be *integrable* if

$$[T^{1,0}(M), T^{1,0}(M)] \subset T^{1,0}(M), \tag{1.2}$$

which means that the bracket of any two sections of $T^{1,0}(M)$ is again a section of $T^{1,0}(M)$. For the theory of general integrable structures, we refer the readers to [BCH].

We consider the exterior algebra of differential forms with complex coefficients:

$$\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^{2m} \ ,$$

where Ω^0 is the ring of smooth complex-valued functions and Ω^r ($r = 1, \dots, 2m$) is the module over Ω^0 of complex-valued smooth r-forms on M.

Definition 1.1. A subalgebra $\mathcal{I}$ of Ω^* is called an *algebraic ideal* if the following conditions hold:

i) $\mathcal{I} \wedge \Omega^* \subset \mathcal{I}$,
ii) if $\phi = \sum_{r=0}^{2m} \phi_r \in \mathcal{I}$, where $\phi_r \in \Omega^r$, then each component $\phi_r \in \mathcal{I}$ (homogeneity condition).

The homogeneity condition implies that $\mathcal{I}$ is a two-sided ideal, that is, $\Omega^* \wedge \mathcal{I} \subset \mathcal{I}$.

In this paper we consider ideals generated by finitely many complex-valued functions and finitely many 1-forms. Let $\rho = (\rho^1, \dots, \rho^d)$ and $\phi = (\phi^1, \dots, \phi^q)$ be a system of functions and 1-forms, respectively. We

denote by $\mathcal{I}(\rho,\phi)$, or simply by (ρ,ϕ), the algebraic ideal generated by ρ and ϕ, which is the set of all elements of Ω^* of the form

$$\sum_{\alpha=1}^{d}\rho^\alpha\omega^\alpha+\sum_{k=1}^{q}\phi^k\wedge\psi^k,\qquad \text{for some } \omega^\alpha,\psi^k\in\Omega^*.$$

For two elements α and β of Ω^*

$$\alpha\equiv\beta\quad \mod (\rho,\phi)$$

means that $\alpha-\beta\in\mathcal{I}(\rho,\phi)$.

The integrability condition (1.2) can be written as

$$[Z_j,Z_k]\in\Gamma(T^{1,0}(M)),\qquad \text{for all } j,k=1,\dots,m, \tag{1.3}$$

where Γ denotes the set of all smooth sections. (1.3) is equivalent to

$$d\theta^\ell\equiv 0,\qquad \mod(\theta),\qquad \text{for all } \ell=1,\dots,m,$$

where $\theta=(\theta^1,\dots,\theta^m)$.

Theorem 1.2. *(Newlander-Nirenberg [NN]) Let (M^{2m},J) be a C^∞ almost complex manifold. If J is integrable then there exist m independent J-holomorphic functions.*

The converse is also true, which is rather trivial. Now we fix notations: For any sub-bundle $I\subset(T^*M)^{1,0}$ we denote by $\underline{I}$ the module over Ω^0 of smooth sections of I and by (I) the algebraic ideal of Ω^* generated by the smooth sections of I. By using Theorem 1.2 and the Frobenius theorem the following was proved in [Tr]:

Theorem 1.3. *Suppose that T' is a sub-bundle of $(T^*M)^{1,0}$ of rank q $(q<m)$ and that T' is closed, that is, $d\underline{T'}\subset(T')$. Then there exist q independent J-holomorphic functions $\zeta^1,\dots,\zeta^q$ whose differentials $d\zeta^1,\dots,d\zeta^q$ span T'.*

The problem of determining conditions for the existence of J-holomorphic functions on almost complex manifolds has been studied in [M1,M2] by studying the involutivity of the Nijenhuis bundle. Criteria for the existence of J-holomorphic mappings into another almost complex manifolds are given in [Kr] in terms of Nijenhuis tensors and their generalizations. The following theorem is found in [HK].

Theorem 1.4. *Let M^{2m} $(m\geq 2)$ be a C^∞ manifold with C^∞ almost complex structure J. Let $(T^*M)^{1,0}$ be the bundle of $(1,0)$-forms. Then there exist a sequence of sub-bundles $(T^*M)^{1,0}:=I^{(0)}\supset I^{(1)}\supset I^{(2)}\supset\cdots$ and a non-negative integer ν such that for $k=0,1,2,\dots,$*

(i) $I^{(k+1)}\subsetneq I^{(k)}$, *if* $k<\nu$.
(ii) $I^{(k+1)}=I^{(k)}$, *if* $k\geq\nu$.

(iii) $d\underline{I}^{(k+1)} \equiv 0$, *mod* $I^{(k)}$,

under a generic assumption in each step of the construction of the sequence. Moreover, a function f is J-holomorphic if and only if $df \in I^{(\nu)}$, thus the number of independent J-holomorphic functions is equal to the rank of $I^{(\nu)}$.

Definition 1.5. The integer ν of Theorem 1.4 is called the *type* of the almost complex structure J. We also say that the Pfaffian system $(\theta^1, \dots, \theta^m)$ has derived length ν.

Proof of Theorem 1.4:
We shall find the largest closed sub-bundle of $(T^*M)^{1,0}$ starting with $I = I^{(0)} = (T^*M)^{1,0}$: The exterior derivative $d : \underline{\mathrm{I}} \to \Omega^2$ is not a module homomorphism, but composition with the projection

$$\underline{\mathrm{I}} \xrightarrow{d} \Omega^2 \xrightarrow{\pi} \Omega^2/(I)$$

is an Ω^0-module homomorphism. Let $\delta = \pi \circ d$. Consider the submodule $\underline{\mathrm{I}}^{(1)} := ker\ \delta$ of $\underline{\mathrm{I}}$. We assume that $\underline{\mathrm{I}}^{(1)}$ has constant rank on M, hence defines a sub-bundle $I^{(1)}$ of $(T^*M)^{1,0}$. We have a short exact sequence of Ω^0-modules

$$0 \to \underline{\mathrm{I}}^{(1)} \to \underline{\mathrm{I}} \xrightarrow{\delta} d\underline{\mathrm{I}}/(I) \to 0.$$

The sub-bundle $I^{(1)}$ is called the first derived system of $(T^*M)^{1,0}$. Assuming that $\underline{\mathrm{I}}^{(k-1)}$ has constant rank, we define inductively k-th derived system $I^{(k)}$ by

$$0 \to \underline{\mathrm{I}}^{(k)} \to \underline{\mathrm{I}}^{(k-1)} \xrightarrow{\delta} d\underline{\mathrm{I}}^{(k-1)}/(I^{(k-1)}) \to 0.$$

Let ν be the smallest integer with $I^{(\nu)} = I^{(\nu+1)}$. Then we have sequence of sub-bundles

$$(T^*M)^{1,0} := I := I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(\nu-1)} \supset I^{(\nu)}. \tag{1.4}$$

Notice that $d\underline{\mathrm{I}}^{(\nu)} \subset (I^{(\nu)})$, that is, $I^{(\nu)}$ is closed. Assume that $\underline{\mathrm{I}}^{(\nu)}$ has constant rank q. Then by Theorem 1.3 there exist independent J-holomorphic functions $\zeta^1, \dots, \zeta^q$, which completes the proof of Theorem 1.4.

The idea of Theorem 1.4 came from the theory of first integrals for Pfaffian systems due to E. Cartan and R. Gardner [Gar], which is a real version of Theorem 1.4. A generalized notion of the first integral has been used in [AH] and in [HP]. Our standard reference for the

theory of Pfaffian system is [BC3G]. In this paper we need a notion of J-holomorphicity on the zero set that we define as follows:

Definition 1.6. A system of complex-valued functions $\zeta = (\zeta^1, \dots, \zeta^d)$ is said to be *J-holomorphic on the zero set* if for each $\alpha = 1, \dots, d$, $(\bar{Z}_j \zeta^\alpha)(x) = 0$, $j = 1, \dots, m$, for all x with $\zeta(x) = 0$, or equivalently, if

$$d\zeta^\alpha \equiv 0, \quad \mod (\zeta, \bar{\zeta}, \theta). \tag{1.5}$$

Assuming further that θ^j are dual to Z_k, that is,

$$\theta^j(Z_k) = \delta^j_k,$$

we define ∂f and $\bar{\partial} f$ for any complex-valued function f by

$$\partial f := \sum_{j=1}^m (Z_j f)\theta^j, \qquad \bar{\partial} f := \sum_{j=1}^m (\bar{Z}_j f)\bar{\theta}^j.$$

Then we have

$$df = \partial f + \bar{\partial} f.$$

We may write (1.5) as

$$\bar{\partial}\zeta^\alpha \equiv 0, \quad \mod (\zeta, \bar{\zeta}), \quad \text{for each } \alpha = 1, \dots, d.$$

2. J-invariant submanifolds

A submanifold $N \subset M$ is said to be J-invariant if $JT_xN = T_xN$, at every point $x \in N$. J-invariant submanifolds are even dimensional. In this section we shall discuss the properties of a system of real-valued functions

$$\rho = (\rho^1, \dots, \rho^{2d})$$

that defines a J-invariant submanifold N. The system ρ shall be called non-degenerate if

$$d\rho^1 \wedge \cdots \wedge d\rho^{2d} \neq 0.$$

Given a set of finitely many differential 1-forms $\{\phi^1, \phi^2, \dots, \}$ we shall mean by the rank at $x \in M$ the number of independent 1-forms at x.

Now we consider a submanifold N^{2n} of (M^{2m}, J) locally defined as the common zero set of a non-degenerate set of real-valued functions $\rho^1, \dots, \rho^{2d}$ with $d = m - n$.

Proposition 2.1. *Suppose that $(\rho^1, \dots, \rho^{2d})$ is a non-degenerate set of real-valued functions on a neighborhood of a point x of (M^{2m}, J) with $d \le m$. Then we have*

$$d \le \operatorname{rank}(\partial\rho^1, \dots, \partial\rho^{2d}) \le 2d\ .$$

Proof. Consider

$$\begin{aligned} d\rho^1 \wedge \cdots \wedge d\rho^{2d} &= (\partial\rho^1 + \bar\partial\rho^1) \wedge \cdots \wedge (\partial\rho^{2d} + \bar\partial\rho^{2d}) \\ &= (\partial\rho^1 \wedge \cdots \wedge \partial\rho^{2d}) + \text{mixed terms} + (\bar\partial\rho^1 \wedge \cdots \wedge \bar\partial\rho^{2d})\ , \end{aligned} \tag{2.1}$$

where "mixed terms" means those terms that contain both $\partial\rho^\alpha$'s and $\bar\partial\rho^\alpha$'s. If $\operatorname{rank}(\partial\rho^1, \dots, \partial\rho^{2d}) \le d-1$ then each term in the last line of (2.1) contains either $\partial\rho^\alpha$'s more than d times or $\bar\partial\rho^\alpha$'s more than d times. Hence, each term of the last line of (2.1) is zero at x, which contradicts to the non-degeneracy condition. □

Proposition 2.2. *Let u be a C^∞ complex-valued function on M and $X \in TM$. Then*

$$\begin{aligned} \partial u(X) &= \frac{1}{2}\{du(X) - \sqrt{-1}du(JX)\} \quad \textit{and} \\ \bar\partial u(X) &= \frac{1}{2}\{du(X) + \sqrt{-1}du(JX)\}. \end{aligned}$$

Proof. Since ∂u annihilates any $(0,1)$-vector, we have

$$\begin{aligned} \partial u(X) &= \partial u(X^{1,0} + X^{0,1}) = \partial u(X^{1,0}) = du(X^{1,0}) \\ &= \frac{1}{2}du\{X - \sqrt{-1}JX\} = \frac{1}{2}\{du(X) - \sqrt{-1}du(JX)\}\ . \end{aligned}$$

We prove the second equality similarly. □

Theorem 2.3. *Let N^{2n} be a submanifold of (M^{2m}, J) given as a common zero set of a non-degenerate system of real-valued functions $\rho^1, \dots, \rho^{2d}$ with $d = m - n$. Let $T^{1,0}N = \{X - \sqrt{-1}JX : X \in TN \cap JTN\}$ and $T^{0,1}N = \{X + \sqrt{-1}JX : X \in TN \cap JTN\}$. Then the following are equivalent:*

(i) *N is J-invariant.*
(ii) *$T^{1,0}_xN$ and $T^{0,1}_xN$ have complex dimension n for each $x \in N$.*
(iii) *$\operatorname{rank}(\partial\rho^1, \dots, \partial\rho^{2d})(x) = d$ for each $x \in N$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that N is J-invariant. Then it is easy to see that there exist linearly independent real vector fields

$$X_1, JX_1, \ldots, X_n, JX_n$$

that are tangent to N. Thus $2n$ complex vectors $X_k^{1,0} := \frac{1}{2}(X_k - \sqrt{-1}JX_k)$ and $X_k^{0,1} := \frac{1}{2}(X_k + \sqrt{-1}JX_k)$, $k = 1, \ldots, n$, are linearly independent and tangent to N, which implies (ii).

(ii) $\Rightarrow$ (iii): Suppose that for each $x \in N$, $T_x^{1,0}N$ has complex dimension n. Since

$$\begin{aligned} T^{1,0}N &= \{Z \in T^{1,0}M : d\rho^\alpha(Z) = \partial\rho^\alpha(Z) = 0,\ \alpha = 1, \ldots, 2d\} \\ &= \bigcap_{\alpha=1}^{2d} (\operatorname{Ker} \partial\rho^\alpha \cap T^{1,0}M) \\ &= \left(\bigcap_{\alpha=1}^{2d} (\operatorname{Ker} \partial\rho^\alpha) \right) \cap T^{1,0}M \end{aligned}$$

has a fibre of complex dimension n at each point $x \in N$, it follows that $(\partial\rho^1, \ldots, \partial\rho^{2d})$ has rank $m - n = d$ at x.

(iii) $\Rightarrow$ (i): Since $T_x^{1,0}M$ is of complex dimension m and $(\partial\rho^1, \ldots, \partial\rho^{2d})$ has rank d, the intersection of the null spaces of $\partial\rho^\alpha : T_x^{1,0}M \to \mathbb{C}$, $\alpha = 1, \ldots, 2d$, is of complex dimension $m - d = n$, and therefore, contains linearly independent vectors $X_1^{1,0}, \ldots, X_n^{1,0}$, where $X_k^{1,0} = \frac{1}{2}(X_k - \sqrt{-1}JX_k)$ for some real vector X_k. Then for each $\alpha = 1, \ldots, 2d$ and each $k = 1, \ldots, n$ we have by Proposition 2.2

$$0 = \partial\rho^\alpha(X_k^{1,0}) = \partial\rho^\alpha(X_k) = \frac{1}{2}(d\rho^\alpha(X_k) - \sqrt{-1}d\rho^\alpha(JX_k)) \ ,$$

which implies that $d\rho^\alpha(X_k) = 0$ and $d\rho^\alpha(JX_k) = 0$ since ρ^α's are real-valued functions. Therefore, $\{X_k, JX_k : k = 1, \ldots, n\}$ are tangent to N. Since $\{X_k^{1,0}, k = 1, \ldots, n\}$ are independent, the set of vectors X_j, JX_j $(j = 1, \ldots, n)$ forms a J-invariant basis for T_xN. Therefore, N is J-invariant. □

The J-invariance of submanifolds has been studied in [HL]. As for the special cases of real codimension 2 ($d = 1$) we have the following

Corollary 2.4. *Let (s, t) be a non-degenerate set of real-valued functions of (M^{2m}, J) and let $N^{2(m-1)}$ be the common zero set of s and t. Then N is J-invariant if and only if*

$$\partial s \wedge \partial t \equiv 0, \quad \textit{mod } (s, t). \tag{2.2}$$

Proof. In the case of $d = 1$ in Proposition 2.3 the rank condition iii)

$$\text{rank } (\partial s, \partial t)(x) = 1 \quad \text{for each } x \in N$$

can be written as (2.2). □

Theorem 2.5. *Let (M^{2m}, J) be an almost complex manifold. A submanifold N^{2n} of real codimension $2d$, where $d = m - n$, is J-invariant if and only if N is the common zero set of a set of complex-valued functions $\zeta = (\zeta^1, \dots, \zeta^d)$ that are J-holomorphic on the zero set.*

Proof. Suppose that N is a J-invariant submanifold of real codimension $2d$. Let $(\rho^1, \dots, \rho^{2d})$ be a non-degenerate set of real-valued functions whose common zero set is N. Since $(\partial\rho^1, \dots, \partial\rho^{2d})$ has rank d by Theorem 2.3, we may assume that $\partial\rho^1 \wedge \cdots \wedge \partial\rho^d \neq 0$. Then for each $\alpha = 1, \dots, d$, $\partial\rho^{d+\alpha}$ is a linear combination of $(\partial\rho^1, \dots, \partial\rho^d)$, or equivalently,

$$\begin{bmatrix} \bar\partial\rho^{d+1} \\ \vdots \\ \bar\partial\rho^{2d} \end{bmatrix} = A \begin{bmatrix} \bar\partial\rho^1 \\ \vdots \\ \bar\partial\rho^d \end{bmatrix}, \tag{2.3}$$

for some invertible matrix $A = (a^\alpha_\beta)$ of smooth functions. Define $\zeta = (\zeta^1, \dots, \zeta^d)$ by

$$\begin{bmatrix} \zeta^1 \\ \vdots \\ \zeta^d \end{bmatrix} = \begin{bmatrix} \rho^{d+1} \\ \vdots \\ \rho^{2d} \end{bmatrix} - A \begin{bmatrix} \rho^1 \\ \vdots \\ \rho^d \end{bmatrix}.$$

By (2.3) we have

$$\bar\partial\zeta^\alpha \equiv 0, \quad \text{mod } (\rho), \quad \text{for each } \alpha = 1, \cdots, d.$$

Since $\mathcal{I}(\rho) = \mathcal{I}(\zeta, \bar\zeta)$ it follows that the set of complex-valued functions $\zeta = (\zeta^1, \dots, \zeta^d)$ is J-holomorphic on the zero set. Conversely, suppose that $\zeta = (\zeta^1, \dots, \zeta^d)$ with $d\zeta^1 \wedge \cdots \wedge d\zeta^d \neq 0$ is J-holomorphic on the zero set. Let $\zeta^\alpha = s^\alpha + it^\alpha$. Then $\bar\partial\zeta^\alpha = \bar\partial s^\alpha + i\bar\partial t^\alpha \equiv 0$, mod $(\zeta, \bar\zeta)$, which implies

$$\partial t^\alpha \equiv -i\partial s^\alpha, \quad \text{mod } (\zeta, \bar\zeta). \tag{2.4}$$

Hence the rank of $(\partial s^1, \partial t^1, \dots, \partial s^d, \partial t^d)$ is at most d. On the other hand, since

$$\begin{aligned} d\zeta^\alpha &\equiv \partial\zeta^\alpha \quad \text{mod } (\zeta, \bar\zeta) \\ &\equiv 2\partial s^\alpha, \quad \text{by (2.4)}, \end{aligned}$$

we have

$$\begin{aligned} 2^d \partial s^1 \wedge \cdots \wedge \partial s^d &\equiv \partial \zeta^1 \wedge \cdots \wedge \partial \zeta^d \quad \mod (\zeta, \bar{\zeta}) \\ &\equiv d\zeta^1 \wedge \cdots \wedge d\zeta^d, \quad \mod (\zeta, \bar{\zeta}) \\ &\neq 0. \end{aligned}$$

Hence $(\partial s^1, \partial t^1, \cdots, \partial s^d, \partial t^d)$ has rank d. Then it follows from Theorem 2.3 that N is J-invariant. □

3. Non-linearly perturbed Cauchy-Riemann equations

Let $z = (z^1, \ldots, z^n) \in \mathbb{C}^n, n \geq 1$, $z^j = x^j + \sqrt{-1} y^j$. In this section we discuss the local solvability of the quasi-linear Cauchy-Riemann equations for one unknown function w:

$$\frac{\partial w}{\partial \bar{z}^j} + \sum_{k=1}^{n} A_j^k(z, \bar{z}, w, \bar{w}) \frac{\partial w}{\partial z^k} = B_j(z, \bar{z}, w, \bar{w}), \quad j = 1, \ldots, n, \tag{3.1}$$

where A_j^k and B_j are complex-valued C^∞ functions defined on a neighborhood of the origin of $\mathbb{C}^{n+1} = \{(z, w)\}$. We shall assume that the coefficients A_j^k are sufficiently small, more precisely, we assume

$$|\det(A_j^k)| < 1. \tag{3.2}$$

We consider a system $(Z_1, \ldots, Z_{n+1})$ of complex vector fields on an open neighborhood of origin of $\mathbb{C}^{n+1} = \{(z, w)\}$ whose complex conjugates are given by

$$\begin{aligned} \bar{Z}_j &= \frac{\partial}{\partial \bar{z}^j} + \sum_{k=1}^{n} A_j^k \frac{\partial}{\partial z^k} + B_j \frac{\partial}{\partial w}, \quad j = 1, \ldots, n, \\ \bar{Z}_{n+1} &= \frac{\partial}{\partial \bar{w}}. \end{aligned} \tag{3.3}$$

Then $\bar{Z}_1, \ldots, \bar{Z}_{n+1}$ are annihilated by the following set of independent 1-forms:

$$\begin{aligned} \theta^\alpha &= dz^\alpha - \sum_{j=1}^{n} A_j^\alpha d\bar{z}^j, \quad \alpha = 1, \ldots, n, \\ \theta^{n+1} &= dw - \sum_{j=1}^{n} B_j d\bar{z}^j. \end{aligned} \tag{3.4}$$

It is easy to check (3.2) implies that the functions A_j^k and B_j define an almost complex structure J on $\mathbb{C}^{n+1}$, for which $\bar{Z}_j$'s are $(0,1)$-vector fields, or equivalently, θ^α's are $(1,0)$-forms.

Proposition 3.1. *For each $j,k=1,\dots,n$, let A_j^k and B_j be C^∞ complex-valued functions defined on a neighborhood of the origin of $\mathbb{C}^{n+1}=\{(z,w)\}$ that satisfy (3.2). Let J be the almost complex structure whose $(0,1)$-vectors and $(1,0)$-forms are given by (3.3)-(3.4). Suppose that J has type ν and that $I^{(\nu)}$ has rank q. Then there exist q independent J-holomorphic functions $\zeta^1,\dots,\zeta^q$. A function ζ is a J-holomorphic function if and only if ζ is holomorphic in the variables $\zeta^1,\dots,\zeta^q$.*

Proof. The first part of the conclusion follows from Theorem 1.3 and Theorem 1.4. To prove the second assertion, suppose that ζ is J-holomorphic. Since $d\zeta\in \underline{I}^{(\nu)}$ we have

$$d\zeta=\sum_{\alpha=1}^{q} a_\alpha d\zeta^\alpha, \tag{3.5}$$

for some C^∞ functions a_α. Without loss of generality assume

$$(z^1,\dots,z^p,\zeta^1,\dots,\zeta^q),\qquad p+q=n+1,$$

are independent functions, so that they serve as C^∞ local coordinates of $\mathbb{C}^{n+1}$. Then (3.5) implies

$$\frac{\partial\zeta}{\partial z^j}=\frac{\partial\zeta}{\partial\bar{z}^j}=\frac{\partial\zeta}{\partial\bar{\zeta}^\alpha}=0,$$

for all $j=1,\dots,p$ and $\alpha=1,\dots,q$, which means that ζ is holomorphic in $(\zeta^1,\dots,\zeta^q)$. Conversely, if ζ is a function in the variables $\zeta^1,\dots,\zeta^q$, then we have

$$d\zeta\in\mathcal{I}(d\zeta^1,\dots,d\zeta^q)=I^{(\nu)}.$$

Therefore, ζ is J-holomorphic. □

Theorem 3.2. *Under the same hypotheses as in Proposition 3.1 let ζ be a J-holomorphic function with $\frac{\partial\zeta}{\partial w}\neq 0$. Then*

$$\zeta=\textit{constant} \tag{3.6}$$

is an implicit solution of (3.1).

Proof. Since $\frac{\partial\zeta}{\partial w} \neq 0$ and $\frac{\partial\zeta}{\partial\bar{w}} = 0$, by implicit function theorem we can solve (3.6) for w to have $w = f(z,\bar{z})$, that is,

$$\zeta(z,\bar{z},f(z,\bar{z}),\overline{f(z,\bar{z})}) = 0. \tag{3.7}$$

Differentiating (3.7) in $\bar{z}^j$ and in z^k, respectively, we obtain

$$\begin{aligned} &\frac{\partial\zeta}{\partial\bar{z}^j} + \frac{\partial\zeta}{\partial w}\frac{\partial f}{\partial\bar{z}^j} = 0 \\ &\frac{\partial\zeta}{\partial z^k} + \frac{\partial\zeta}{\partial w}\frac{\partial f}{\partial z^k} = 0, \quad j,k = 1,\dots,n. \end{aligned} \tag{3.8}$$

Since ζ is J-holomorphic we have $L_j\zeta = 0, j = 1,\dots,n$, namely

$$\frac{\partial\zeta}{\partial\bar{z}^j} + \sum_{k=1}^{n} A_j^k \frac{\partial\zeta}{\partial z^k} + B_j\frac{\partial\zeta}{\partial w} = 0, \quad j = 1,\dots,n. \tag{3.9}$$

From (3.8) and (3.9) it follows that

$$-\frac{\partial\zeta}{\partial w}\left(\frac{\partial f}{\partial\bar{z}^j} + \sum_{k=1}^{n} A_j^k\frac{\partial f}{\partial z^k} - B_j\right) = 0, \tag{3.10}$$

which implies the conclusion. □

Theorem 3.3. *Let $(\bar{Z}_1,\dots,\bar{Z}_{n+1})$ and $(\theta^1,\dots,\theta^{n+1})$ be the same as in (3.3)-(3.4) and let J be the almost complex structure with $(0,1)$-vectors Z_j (or equivalently, $(1,0)$-forms θ^j). Then there exists a solution $w = f(z,\bar{z})$ of (3.1) if and only if there exists a function $\zeta(z,\bar{z},w,\bar{w})$ with $\frac{\partial\zeta}{\partial w} \neq 0$ which is J-holomorphic on the zero set.*

Proof. Suppose that $w = f(z,\bar{z})$ is a solution of (3.1). Then

$$\zeta(z,\bar{z},w,\bar{w}) := f(z,\bar{z}) - w$$

satisfies $\bar{Z}_j\zeta \equiv 0$, mod $(\zeta,\bar{\zeta})$, for all $j = 1,\dots,n+1$. Conversely, suppose that $\zeta(z,\bar{z},w,\bar{w})$ with $\frac{\partial\zeta}{\partial w} \neq 0$ is J-holomorphic on the zero set. Since $\frac{\partial\zeta}{\partial\bar{w}} = 0$ on the zero set, by the implicit function theorem we can solve $\zeta = 0$ for w, to obtain $w = f(z,\bar{z})$, that is,

$$\zeta(z,\bar{z},f(z,\bar{z}),\overline{f(z,\bar{z})}) = 0. \tag{3.11}$$

Then by differentiating (3.11) with respect to $\bar{z}^j$ and z^k, respectively, and restricting to the zero set of ζ we have (3.8)-(3.10) and the proof is same as that of Theorem 3.2. □

For the existence of solutions of (3.1) the coefficients A_j^k and B_j must satisfy certain conditions. To discuss this we first define smooth functions T_{ij}^α by setting

$$d\theta^\alpha \equiv \sum_{1\le i<j\le n+1} T_{ij}^\alpha d\bar{z}^i \wedge d\bar{z}^j, \quad \mathrm{mod}\ (\theta), \quad \alpha = 1, \dots, n+1, \tag{3.12}$$

where $z^{n+1} = w$, $\bar{z}^{n+1} = \bar{w}$. Arranging the pairs (ij) with $i < j$ in lexicographical order, we write (3.12) in matrices as

$$\begin{bmatrix} d\theta^1 \\ \vdots \\ d\theta^{n+1} \end{bmatrix} \equiv \underbrace{\begin{bmatrix} T_{12}^1 & T_{13}^1 & \cdots & T_{n,n+1}^1 \\ \vdots & \vdots & & \vdots \\ T_{12}^{n+1} & T_{13}^{n+1} & \cdots & T_{n,n+1}^{n+1} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} d\bar{z}^1 \wedge d\bar{z}^2 \\ d\bar{z}^1 \wedge d\bar{z}^3 \\ \vdots \\ d\bar{z}^n \wedge d\bar{z}^{n+1} \end{bmatrix}, \ \mathrm{mod}\ (\theta).$$

The matrix $\mathcal{T}$ of size $(n+1) \times \binom{n+1}{2}$ shall be called the torsion of the Pfaffian system (3.4).

If $\mathcal{T}$ has rank zero, that is, if all T_{ij}^α are zero, this is the case $I = I^{(1)}$, and I is closed. Then by Theorem 1.2 there exist $n+1$ independent J-holomorphic functions.

Theorem 3.4. *Suppose that there exist J-holomorphic functions $\zeta^1, \dots, \zeta^q$, $q \le n+1$, with $d\zeta^1 \wedge \cdots \wedge d\zeta^q \neq 0$. Then $\mathcal{T}$ has rank at most $(n+1)-q$.*

Proof. For each $\lambda = 1, \dots, q$, let

$$d\zeta^\lambda = \sum_{\alpha=1}^{n+1} a_\alpha^\lambda \theta^\alpha, \tag{3.13}$$

for some functions a_α^λ. Applying d to (3.13) we have

$$0 \equiv \sum_\alpha a_\alpha^\lambda d\theta^\alpha, \quad \mathrm{mod}\ (\theta), \tag{3.14}$$

for each $\lambda = 1, \dots, q$. Substituting (3.12) in (3.14) we have

$$0 \equiv \sum_{\alpha=1}^{n+1} a_\alpha^\lambda \sum_{1\le i<j\le n+1} T_{ij}^\alpha d\bar{z}^i \wedge d\bar{z}^j, \quad \mathrm{mod}\ (\theta),$$

which is written in matrices as

$$\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} \equiv \underbrace{\begin{bmatrix} a_1^1 & \cdots & a_{n+1}^1\\ \vdots & & \vdots\\ a_1^q & \cdots & a_{n+1}^q\end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} T_{12}^1 & T_{13}^1 & \cdots & T_{n,n+1}^1\\ \vdots & \vdots & & \vdots\\ T_{12}^{n+1} & T_{13}^{n+1} & \cdots & T_{n,n+1}^{n+1}\end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} d\bar{z}^1\wedge d\bar{z}^2\\ d\bar{z}^1\wedge d\bar{z}^3\\ \vdots\\ d\bar{z}^n\wedge d\bar{z}^{n+1}\end{bmatrix},$$

mod (θ).

Since $d\zeta^1\wedge\cdots\wedge d\zeta^q\neq 0$, (3.13) implies that $\mathcal{A}$ has maximal rank q. Each row of $\mathcal{A}$ gives a linear relation among the rows of $\mathcal{T}$. Therefore, $\mathcal{T}$ has rank at most $(n+1)-q$. □

We construct the sequence (1.4) of sub-bundles as follows: An element $\phi=\sum_{\alpha=1}^{n+1}a_\alpha\theta^\alpha$ of I belongs to $I^{(1)}$ if and only if $(a_1,\dots,a_{n+1})$ is a null vector of the matrix $\mathcal{T}$, because

$$\begin{aligned} d\phi &\equiv \sum_{\alpha=1}^{n+1} a_\alpha d\theta^\alpha, \quad \text{mod } (\theta)\\ &\equiv \sum_{1\le i<j\le n+1}\sum_{\alpha=1}^{n+1} a_\alpha T_{ij}^\alpha d\bar{z}^i\wedge d\bar{z}^j, \quad \text{mod } (\theta)\end{aligned}$$

is zero if and only if $(a_1,\dots,a_{n+1})$ is a null vector of $\mathcal{T}$, that is,

$$\sum_{\alpha=1}^{n+1} a_\alpha T_{ij}^\alpha = 0, \quad \text{for all pairs } (ij).$$

Inductively, let $\phi=(\phi^1,\dots,\phi^r)$ be a set of generators of $I^{(k)}$. Then a 1-form $\psi=\sum_{\beta=1}^r b_\beta\phi^\beta$ is an element of $I^{(k+1)}$ if and only if $(b_1,\dots,b_r)$ is a null vector of the torsion matrix of the Pfaffian system ϕ. Now suppose that (3.4) has derived length ν. In the construction of $I^{(\nu)}$ the coefficients A_j^k, B_j in (3.4) are differentiated up to ν times and then the condition

$$dI^{(\nu)}\subset (I^{(\nu)}) \tag{3.15}$$

raises the order of the derivatives by one. Thus we have

Proposition 3.5. *Let J be the almost complex structure on $\mathbb{C}^{n+1}$ whose $(1,0)$-forms are given in (3.4). Then its type condition is a system of partial differential equations on (A_j^k, B_j) : Condition (3.15) of being type ν is a PDE system of order $\nu+1$. If J has type ν and $I^{(\nu)}$ has*

rank q, then there exists a complex q-parameter family of solutions of (3.1).

Summarizing our previous discussions in Theorem 3.3, Theorem 2.5 and Corollary 2.4 we have

Theorem 3.6. *Given a system of quasi-linear Cauchy-Riemann equations (3.1) with coefficients satisfying (3.2), let J be the almost complex structure on $\mathbb{C}^{n+1} = \{(z,w)\}$, $z = (z^1,\dots,z^n)$, $n \geq 2$, with $(1,0)$-forms (3.4). Then (3.1) has a solution if and only if there exists a non-degenerate system of real-valued functions (s,t) having the following properties:*
i) The determinant of any square submatrices of maximal size of $\mathcal{T}$ is zero modulo (s,t).
ii) $\partial s \wedge \partial t \equiv 0, \quad mod\ (s,t)$.

Condition ii) means that the common zero set of s and t is J-invariant. Condition i) means that we construct s and t by finding a non-degenerate set of real-valued functions that generates an ideal that the determinants of $n \times n$ submatrices of the torsion belong to.

4. Cases of several unknown functions

Our arguments of the previous section can easily be generalized to the cases of several unknown functions. Let $z = (z^1,\dots,z^n) \in \mathbb{C}^n, n \geq 1$, $z^j = x^j + \sqrt{-1}y^j$. We consider the system of quasi-linear Cauchy-Riemann equations for $w = (w^1,\dots,w^d)$, $d \geq 2$:

$$\frac{\partial w^\alpha}{\partial \bar{z}^j} + \sum_{k=1}^{n} A_j^k(z,\bar{z},w,\bar{w})\frac{\partial w^\alpha}{\partial z^k} = B_j^\alpha(z,\bar{z},w,\bar{w}), \tag{4.1}$$

for each $j = 1,\dots,n$ and $\alpha = 1,\dots,d$, where A_j^k and B_j^α are complex-valued C^∞ functions defined on a neighborhood of the origin of $\mathbb{C}^{n+d} = \{(z,w)\}$ satisfying (3.2). We consider a system $(Z_1,\dots,Z_{n+d})$ of complex vector fields on an open neighborhood of origin of $\mathbb{C}^{n+d} = \{(z,w)\}$

whose complex conjugates are given by

$$\begin{aligned} \bar{Z}_j &= \frac{\partial}{\partial \bar{z}^j} + \sum_{k=1}^{n} A_j^k \frac{\partial}{\partial z^k} + \sum_{\alpha=1}^{d} B_j^\alpha \frac{\partial}{\partial w^\alpha}, \quad j = 1, \dots, n, \\ \bar{Z}_{n+\beta} &= \frac{\partial}{\partial \bar{w}^\beta}, \quad \beta = 1, \dots, d. \end{aligned} \tag{4.2}$$

Then $\bar{Z}_1, \dots, \bar{Z}_{n+d}$ are annihilated by the following set of independent 1-forms:

$$\begin{aligned} \theta^k &= dz^k - \sum_{j=1}^{n} A_j^k d\bar{z}^j, \quad k = 1, \dots, n \\ \theta^{n+\alpha} &= dw^\alpha - \sum_{j=1}^{n} B_j^\alpha d\bar{z}^j, \quad \alpha = 1, \dots, d. \end{aligned} \tag{4.3}$$

The functions A_j^k and B_j^α define an almost complex structure J on $\mathbb{C}^{n+d}$, for which $\bar{Z}_j, \bar{Z}_{n+\beta}$ are $(0,1)$-vector fields, or equivalently, $\theta^k, \theta^{n+\alpha}$ are $(1,0)$-forms. The following is a generalization of Theorem 3.3.

Theorem 4.1. *Let $(\bar{Z}_1, \dots, \bar{Z}_{n+d})$ and $(\theta^1, \dots, \theta^{n+d})$ be the same as in (4.2)-(4.3) and let J be the almost complex structure with $(0,1)$-vectors $\bar{Z}_j$ (or equivalently, $(1,0)$-forms θ^j). Then there exists a set of solutions $w^\alpha = f^\alpha(z, \bar{z})$, $\alpha = 1, \dots, d$, of (4.1) if and only if there exists a set of functions $\zeta^\alpha(z, \bar{z}, w, \bar{w})$, $\alpha = 1, \dots, d$ with $\det\left(\frac{\partial \zeta^\alpha}{\partial w^\beta}\right) \neq 0$ that is J-holomorphic on the zero set.*

Proof. Suppose $w^\alpha = f^\alpha(z, \bar{z})$, $\alpha = 1, \dots, d$, is a solution of (4.1). Let $\zeta^\alpha = f^\alpha(z, \bar{z}) - w^\alpha$. Then for each $j = 1, \dots, n$, and each $\alpha = 1, \dots, d$, we have

$$\begin{aligned} \bar{Z}_j \zeta^\alpha &= \frac{\partial f^\alpha}{\partial \bar{z}^j} + A_j^k \frac{\partial f^\alpha}{\partial z^k} - B_j^\alpha \equiv 0, \quad \mathrm{mod}\ (\zeta, \bar{\zeta}) \\ \bar{Z}_{n+\beta} \zeta^\alpha &= 0. \end{aligned}$$

Therefore, $\zeta = (\zeta^1, \dots, \zeta^d)$ is J-holomorphic on the zero set that satisfies the non-degeneracy condition as in the statement of the theorem. Conversely, suppose that $\zeta = (\zeta^1, \dots, \zeta^d)$ is J-holomorphic on the zero set as in the statement of the theorem. Since $\det\left(\frac{\partial \zeta^\alpha}{\partial w^\beta}\right) \neq 0$ and $\frac{\partial \zeta^\alpha}{\partial \bar{w}^\beta} = 0$, mod $(\zeta, \bar{\zeta})$, we can solve $\zeta = 0$ for $w = (w^1, \dots, w^d)$ by

implicit function theorem, to obtain $w^\alpha = f^\alpha(z,\bar z)$, that is,

$$\zeta^\alpha(z,\bar z, f(z,\bar z), \overline{f(z,\bar z)}) = 0, \quad \alpha = 1,\dots,d, \tag{4.4}$$

where $f = (f^1,\dots,f^d)$. By applying $\frac{\partial}{\partial \bar z^j}$ and $\frac{\partial}{\partial z^k}$, respectively, to (4.4) we have

$$\begin{aligned} &\frac{\partial \zeta^\alpha}{\partial \bar z^j} + \sum_{\beta=1}^d \frac{\partial \zeta^\alpha}{\partial w^\beta}\frac{\partial f^\beta}{\partial \bar z^j} = 0,\\ &\frac{\partial \zeta^\alpha}{\partial z^k} + \sum_{\beta=1}^d \frac{\partial \zeta^\alpha}{\partial w^\beta}\frac{\partial f^\beta}{\partial z^k} = 0. \end{aligned} \tag{4.5}$$

Then for each $j = 1,\dots,n$ and each $\alpha = 1,\dots,d$, we have

$$\begin{aligned} \bar Z_j \zeta^\alpha &= \frac{\partial \zeta^\alpha}{\partial \bar z^j} + \sum_{k=1}^n A_j^k(z,\bar z,w,\bar w)\frac{\partial \zeta^\alpha}{\partial z^k} + \sum_{\beta=1}^d B_j^\beta(z,\bar z,w,\bar w)\frac{\partial \zeta^\alpha}{\partial w^\beta}\\ &\equiv 0, \quad \mathrm{mod}\ (\zeta,\bar\zeta). \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6) we have

$$-\sum_{\beta=1}^d \frac{\partial \zeta^\alpha}{\partial w^\beta}\frac{\partial f^\beta}{\partial \bar z^j} - \sum_{k=1}^n\sum_{\beta=1}^d A_j^k \frac{\partial \zeta^\alpha}{\partial w^\beta}\frac{\partial f^\beta}{\partial z^k} + \sum_{\beta=1}^d B_j^\beta \frac{\partial \zeta^\alpha}{\partial w^\beta} \equiv 0, \quad \mathrm{mod}\ (\zeta,\bar\zeta),$$

which can be written in matrices as

$$\frac{\partial \zeta}{\partial w} E \equiv \begin{bmatrix} 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0 \end{bmatrix}_{d\times n}, \quad \mathrm{mod}\ (\zeta,\bar\zeta)$$

where

$$\frac{\partial \zeta}{\partial w} = \begin{bmatrix} \frac{\partial \zeta^1}{\partial w^1} & \cdots & \frac{\partial \zeta^1}{\partial w^d}\\ \vdots & \ddots & \vdots\\ \frac{\partial \zeta^d}{\partial w^1} & \cdots & \frac{\partial \zeta^d}{\partial w^d} \end{bmatrix}$$

and

$$E = \begin{bmatrix} -\frac{\partial f^1}{\partial \bar z^1} - \sum_{k=1}^n A_1^k \frac{\partial f^1}{\partial z^k} + B_1^1 & \cdots & -\frac{\partial f^1}{\partial \bar z^n} - \sum_{k=1}^n A_n^k \frac{\partial f^1}{\partial z^k} + B_n^1\\ \vdots & \ddots & \vdots\\ -\frac{\partial f^d}{\partial \bar z^1} - \sum_{k=1}^n A_1^k \frac{\partial f^d}{\partial z^k} + B_1^d & \cdots & -\frac{\partial f^n}{\partial \bar z^n} - \sum_{k=1}^n A_n^k \frac{\partial f^d}{\partial z^k} + B_n^d \end{bmatrix}_{d\times n}.$$

Since $\frac{\partial \zeta}{\partial w}$ is invertible E is identically zero on the zero set of ζ, which implies that $w = f(z,\bar z)$ is a solution of (4.1). $\square$

Then the almost complex structure on $\mathbb{C}^{n+d}$ whose $(0,1)$-forms are given by (4.2) has torsion $\mathcal{T}$ of dimension $\binom{(n+d)\times n+d}{2}$. By the same argument as in the previous section we have

Theorem 4.2. *Given a system of quasi-linear Cauchy-Riemann equations (4.1), let J be the almost complex structure on $\mathbb{C}^{n+d} = \{(z,w)\}$, $z = (z^1, \dots, z^n)$, $w = (w^1, \dots, w^d)$, with $(1,0)$-forms (4.3). Then (4.1) has a solution if and only if there exists a non-degenerate system of real-valued functions $(s^1, \dots, s^{2d})$ having the following properties:*
i) Determinant of any $n \times n$ submatrices of the $(n+d) \times \binom{n+d}{2}$ matrix of the torsion $\mathcal{T}$ is zero modulo $(s^1, \dots, s^{2d})$.
ii) $(\partial s^1, \dots, \partial s^{2d})$ has rank d.

Condition ii) means that the common zero set of $s^1, \dots, s^{2d}$ is J-invariant. Condition i) means that we construct $s^1, \dots, s^{2d}$ by finding a non-degenerate set of real-valued functions that generates an ideal that the determinants of $n \times n$ submatrices of the torsion belong to.

5. Quasi-linear Cauchy-Riemann equations in one complex variable

Consider the following equation for a complex-valued function $w = w(z, \bar{z})$:

$$\frac{\partial w}{\partial \bar{z}} + A(z, \bar{z}, w, \bar{w}) \frac{\partial w}{\partial z} = B(z, \bar{z}, w, \bar{w}), \quad |A(z, \bar{z}, w, \bar{w})| < 1. \tag{5.1}$$

This is a determined system of two real equations for two real unknown functions $\Re w$ and $\Im w$. (5.1) is always solvable for the following reason: In $\mathbb{C}^2 = \{(z,w)\}$ we consider complex vector fields

$$\bar{Z}_1 = \frac{\partial}{\partial \bar{z}} + A \frac{\partial}{\partial z} + B \frac{\partial}{\partial w}, \quad \bar{Z}_2 = \frac{\partial}{\partial \bar{w}}$$

and 1-forms that annihilate $\bar{Z}_j, j = 1, 2:$

$$\theta^1 = dz - A d\bar{z}, \quad \theta^2 = dw - B d\bar{z}.$$

An almost-complex structure J on $\mathbb{C}^2$ is uniquely determined by the functions A and B so that $\bar{Z}_j$, $j = 1, 2$, are $(0,1)$-vectors and θ^j, $j = 1, 2$, are $(1,0)$-forms. A fundamental theorem due to A. Nijenhuis and W. B. Woolf [NW] states that for any real tangent vector V of $\mathbb{C}^2$ at the origin there exists a J-holomorphic curve $\gamma(z) = (z, f(z)) : D \to \mathbb{C}^2$, where D is a small open disk centered at the origin in $\mathbb{C}$, satisfying

initial conditions $\gamma(0) = 0$ and $d\gamma(0)(\frac{\partial}{\partial x}) = V$. The graph γ is the zero set of

$$\zeta := f(z, \bar{z}) - w,$$

which is J-holomorphic on its zero set. Thus (5.1) is solvable by Theorem 3.3.

Now we check the type of J. Since

$$\begin{bmatrix} d\theta^1 \\ d\theta^2 \end{bmatrix} \equiv \mathcal{T} d\bar{z} \wedge d\bar{w}, \quad \text{mod } (\theta), \quad \text{where } \mathcal{T} = \begin{bmatrix} A_{\bar{w}} \\ B_{\bar{w}} \end{bmatrix},$$

if $A_{\bar{w}} = B_{\bar{w}} = 0$ the almost complex structure is integrable, and hence by Theorem 1.2 there exist two independent J-holomorphic functions (ζ^1, ζ^2). For any function ζ that is analytic in (ζ^1, ζ^2) such that $\zeta_w \neq 0$

$$\zeta = \text{ constant}$$

is an implicit solution of (5.1). This is the case of type 0. Next, we assume $A_{\bar{w}} \neq 0$. Then $(-B_{\bar{w}}, A_{\bar{w}})$ is a null vector of the torsion $\mathcal{T}$, so that

$$\phi := -B_{\bar{w}}\theta^1 + A_{\bar{w}}\theta^2 \tag{5.2}$$

generates $I^{(1)}$. If

$$d\phi \equiv 0, \quad \text{mod } \phi \tag{5.3}$$

then $I^{(1)}$ is closed. (5.3) is a PDE system of second order for A and B. In summary we have the following table:

rank $\mathcal{T}$	type ν	number of J-holomorphic functions	order of PDEs for A, B	integrability
0	0	2	1	integrable
1	1	1	2	$I^{(1)}$ is closed

Let us consider the following special case of type 1:

$$\frac{\partial w}{\partial \bar{z}} + A(z, \bar{z}, w, \bar{w})\frac{\partial w}{\partial z} = B(\bar{z}, w), \quad A_{\bar{w}} \neq 0. \tag{5.4}$$

Since $B_{\bar{w}} = 0$, from (5.2), $\phi = A_{\bar{w}}\theta^2$ generates $I^{(1)}$. Then computation shows

$$d\phi \equiv 0, \quad \text{mod } (\phi).$$

Thus $I^{(1)}$ has rank 1 and there is a non-degenerate J-holomorphic function ζ. Since $d\zeta \in \mathcal{I}(\theta^2)$ we see that $\zeta_w \neq 0$. Therefore,

$$\zeta = \text{ constant}$$

is a complex 1-parameter family of solutions of (5.4).

6. Examples

Example 6.1. Consider the following system for $w(z^1, \bar{z}^1, z^2, \bar{z}^2)$:

$$\begin{aligned} &\frac{\partial w}{\partial \bar{z}^1} && +w\frac{\partial w}{\partial z^2} = \frac{-2w}{1+\bar{z}^1} \\ &\frac{\partial w}{\partial \bar{z}^2} + \bar{w}\frac{\partial w}{\partial z^1} && +\bar{z}^1\frac{\partial w}{\partial z^2} = 0. \end{aligned} \tag{6.1}$$

Then the associated almost complex structure on $\mathbb{C}^3 = \{(z^1, z^2, w)\}$ has $(1,0)$-forms

$$\begin{aligned} \theta^1 &= dz^1 && - \bar{w}d\bar{z}^2 \\ \theta^2 &= dz^2 && -wd\bar{z}^1 - \bar{z}^1 d\bar{z}^2 \\ \theta^3 &= dw && +\frac{2w}{1+\bar{z}^1}d\bar{z}^1. \end{aligned}$$

Then we have

$$\begin{bmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} d\bar{z}^1 \wedge d\bar{z}^2 \\ d\bar{z}^1 \wedge d\bar{w} \\ d\bar{z}^2 \wedge d\bar{w} \end{bmatrix}, \quad \mathrm{mod}\ (\theta),$$

Hence, $I^{(1)}$ is spanned by θ^3. Since

$$d\theta^3 \equiv 0, \quad \mathrm{mod}\ (\theta^3)$$

this is the case of type 1. There exists a J-holomorphic function ζ. Since $d\zeta$ is a non-zero multiple of θ^3 we see that $\zeta_w \neq 0$. Each level set $\zeta =$ constant is an implicit solution of (6.1).

Example 6.2. Consider the following system for $w(z^1, \bar{z}^1, z^2, \bar{z}^2)$:

$$\begin{aligned} &\frac{\partial w}{\partial \bar{z}^1} && +w\frac{\partial w}{\partial z^2} = \frac{-2w}{1+\bar{z}^1} \\ &\frac{\partial w}{\partial \bar{z}^2} + \bar{w}\frac{\partial w}{\partial z^1} && +\left[z^2 + w(1+\bar{z}^1)\right]\frac{\partial w}{\partial z^2} = 0. \end{aligned} \tag{6.2}$$

Then the associated almost complex structure on $\mathbb{C}^3 = \{(z^1, z^2, w)\}$ has $(1,0)$-forms

$$\begin{aligned}
\theta^1 &= dz^1 && - \bar{w} d\bar{z}^2 \\
\theta^2 &= dz^2 && -w d\bar{z}^1 - \left[z^2 + w(1+\bar{z}^1)\right] d\bar{z}^2 \\
\theta^3 &= dw && + \frac{2w}{1+\bar{z}^1} d\bar{z}^1.
\end{aligned}$$

Let

$$\zeta := z^2 + w(1+\bar{z}^1).$$

Then we see that

$$d\zeta = \theta^2 + (1+\bar{z}^1)\theta^3 + \zeta d\bar{z}^2.$$

Therefore, ζ is J-holomorphic on its zero set. Thus $\zeta = 0$ is an implicit solution of (6.2).

A *pseudo-analytic function* in several complex variables satisfies

$$\frac{\partial w}{\partial \bar{z}^j} = \alpha_j(z) w(z) + \beta_j(z) \overline{w(z)} \quad \text{for } j = 1, \dots, n$$

for some functions $\alpha_j(z)$ and $\beta_j(z)$. See the more details in §7.

Example 6.3. Consider the system (7.4) of pseudo-analytic functions in $\mathbb{C}^2$:

$$\begin{aligned}
\frac{\partial w}{\partial \bar{z}_1} &= \alpha_1(z, \bar{z}) w + \beta_1(z, \bar{z}) \bar{w}, \\
\frac{\partial w}{\partial \bar{z}_2} &= \alpha_2(z, \bar{z}) w + \beta_2(z, \bar{z}) \bar{w}.
\end{aligned}$$

Let $B_j(z, \bar{z}, w, \bar{w}) = \alpha_j(z, \bar{z}) w + \beta_j(z, \bar{z}) \bar{w}$ for $j = 1, 2$. Then the associate almost complex structure on $\mathbb{C}^3 = \{(z^1, z^2, w)\}$ has (1,0)-forms

$$\theta^1 = dz^1, \ \theta^2 = dz^2, \ \theta^3 = dw - B_1 d\bar{z}^1 - B_2 d\bar{z}^2.$$

By applying d to $\theta^1, \theta^2, \theta^3$, we obtain the components of the torsion $\mathcal{T}$ as follows:

$$T^\alpha_{ij} = 0 \quad \text{for } \alpha = 1, 2,$$

and

$$\begin{aligned}
T^3_{12} &= \frac{\partial B_1}{\partial \bar{z}^2} + \frac{\partial B_1}{\partial w} B_2 - \frac{\partial B_2}{\partial \bar{z}^1} - \frac{\partial B_2}{\partial w} B_1 \\
&= \left(\frac{\partial \alpha_1}{\partial \bar{z}^2} - \frac{\partial \alpha_2}{\partial \bar{z}^1} \right) w + \left(\frac{\partial \beta_1}{\partial \bar{z}^2} - \frac{\partial \beta_2}{\partial \bar{z}^1} + \alpha_1 \beta_2 - \alpha_2 \beta_1 \right) \bar{w},
\end{aligned}$$

and

$$T^3_{13} = \frac{\partial B_1}{\partial \bar{w}} = \beta_1,\ T^3_{23} = \frac{\partial B_2}{\partial \bar{w}} = \beta_2.$$

Then $\mathcal{T}$ has rank 0 if and only if $\beta_1 = \beta_2 = 0$ and

$$\frac{\partial \alpha_1}{\partial \bar{z}^2} = \frac{\partial \alpha_2}{\partial \bar{z}^1}.$$

This is the involutive case and there exist three independent pseudo-holomorphic functions for the associated almost complex structure. One of them satisfies $\dfrac{\partial \zeta}{\partial w} \neq 0$, which gives implicit solutions $\zeta =$ constant.

If rank $\mathcal{T} = 1$, then $\phi^1 = \theta^1$ and $\phi^2 = \theta^2$ generates $I^{(1)}$. Since $d\phi^k = 0$ for $k = 1, 2$, this is the case of type 2. However, it is easy to check that there cannot be a function ζ with $\dfrac{\partial \zeta}{\partial w} \neq 0$ that is pseudo-holomorphic on the zero set. Therefore, there are no solutions in this case.

7. Perturbed Cauchy-Riemann equations and overdetermined PDE systems; a brief survey of history

The main object of complex analysis is the family of holomorphic functions $w = w(z)$ which are characterized by the Cauchy-Riemann equations

$$\frac{\partial w}{\partial \bar{z}^j} = 0, \quad j = 1, \ldots, n.$$

It is not surprising that mathematical literature abounds in natural generalizations of the Cauchy-Riemann equations. For the cases $n = 1$ the theories of quasi-conformal functions and pseudo-analytic functions were developed in the mid-20th century. We refer the readers to the references [A1, B2, A2, B3]. A quasi-conformal mapping $w = w(z)$ satisfies the Beltrami equation

$$\frac{\partial w}{\partial \bar{z}} + \mu(z)\frac{\partial w}{\partial z} = 0 \tag{7.1}$$

for some complex-valued Lebesgue measurable function $\mu(z)$ with $|\mu(z)| < 1$. Of central importance in the theory of quasi-conformal mappings in the complex plane is the measurable Riemann mapping theorem [Mor],

which generalizes the Riemann mapping theorem from conformal to quasi-conformal homeomorphisms.

Pseudo-analytic functions [B1, V] are solutions of

$$\frac{\partial w}{\partial \overline{z}} = \alpha(z)w(z) + \beta(z)\overline{w(z)} \tag{7.2}$$

for some functions $\alpha(z)$ and $\beta(z)$. Recall that every harmonic function $\phi(x, y)$ is locally the real part of an analytic function $h(z)$ and the complex gradient $w(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$ is analytic and $w(z) = h'(z)$. On the other hand, a linear elliptic equation for a real-valued function $\phi(x, y)$ of second order with Hölder continuously differentiable coefficients can be transformed into the canonical form

$$\phi_{xx} + \phi_{yy} + A\phi_x + B\phi_y = 0.$$

Then $w := \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$ is a pseudo-analytic function which satisfies the equation (7.2) with $\alpha = -\frac{1}{4}(A + iB)$ and $\beta = -\frac{1}{4}(A - iB)$.

In $\mathbb{C}^n$ with $n \geq 2$, fundamental questions including the Levi problem were solved by means of the inhomogeneous Cauchy-Riemann equations of (p, q)-type (cf. [O, Ko, Hö]), which are generalizations of the $(0, 1)$-type equations

$$\frac{\partial w}{\partial \overline{z}^j} = b_j, \quad \text{for } j = 1, \ldots, n.$$

A multidimensional version of (7.1) is

$$\mathbf{L}_j w := \frac{\partial w}{\partial \overline{z}^j}(z) + \sum_{k=1}^{n} a_j^k(z)\frac{\partial w}{\partial z^k}(z) = 0 \quad \text{for } j = 1, \ldots, n, \tag{7.3}$$

where a_j^k's vanish at the origin. If the coefficients $a_j^k(z)$ are sufficiently smooth, it was shown in [NN] that there exist n linearly independent solutions to (7.3) if and only if $\mathbf{L}_j$ commutes. We state this fundamental result as Theorem 1.2.

A pseudo-analytic function in several complex variables satisfies

$$\frac{\partial w}{\partial \overline{z}^j} = \alpha_j(z)w(z) + \beta_j(z)\overline{w(z)} \quad \text{for } j = 1, \ldots, n \tag{7.4}$$

for some functions $\alpha_j(z)$ and $\beta_j(z)$. The properties of solutions to (7.4) were investigated in [S, Ha]. In particular, it was proved in [S] that (7.4) with $\beta_j(z) = 0$ has the extension property: If $D \subset \mathbb{C}^n$ is a domain and K is a compact subset of D such that $D \setminus K$ is connected, then any solution to (7.4) with $\beta_j(z) = 0$ on $D \setminus K$ extends to D. The extension phenomenon of holomorphic functions in several complex

variables is the special case of this extension property for (7.4) with $\alpha_j(z) = \beta_j(z) = 0$, which was discovered by F. Hartogs [Har] in 1906.

In the 1960s, the theory of overdetermined systems of linear partial differential equations was intensively studied from the algebraic viewpoint based on Spencer complexes [Sp1, Sp2, Sp3]. Quillen [Q], Goldschmidt [Go], MacKichan [Mac] and Sweeney [Sw] investigated the condition for the exactness of the Spencer complexes. The question was whether the Spencer complex is exact if it is elliptic. When the coefficients are real analytic the Spencer complex is exact in the elliptic case. In the C^∞ category, however, one needs to prove an estimate that implies the solvability of the associated Neumann boundary value problem. It turned out that if the linear elliptic overdetermined system satisfies the so-called δ-estimate, then the Neumann problem for the elliptic system is solvable so that the Spencer sequence is exact and hence such a system is locally solvable.

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