Complex Analysis (300.247) Final Examination

December 20, 2001

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1. (10) Evaluate the following integrals:

   a) $\displaystyle\int_{|z|=1} \frac{\cos(\frac{z}{2})-1}{z^2}dz.$

   b) $\displaystyle\int_{|z|=1} \frac{\cos(\frac{z}{2})-1}{z^3}dz.$

2. (10) Find a linear fractional transformation that maps the points $z_1 = 2, \quad z_2 = -1, \quad z_3 = -i$, onto $w_1 = \infty, \quad w_2 = -1, \quad w_3 = -i$, respectively.

3. (10) Find the image of $|z| \leq \frac{1}{2}$ under the transformation $w = \dfrac{z-1}{z+1}$.

4. (10) If $a > e$, show that $e^z = az^n$ has n roots in $|z| < 1$.

5. Evaluate the following integrals:

   a)(15) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}dx.$

   b)(15) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{(x^2+1)(x^2+4)}dx.$

   c)(20) $\displaystyle\int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}dx.$

6. (20) Let $f(z) = \dfrac{1}{z^2(z-2)}$. Find the series representation of $f(z)$ in each of the following regions:

   a) $0 < |z| < 2$.

   b) $|z| > 2$.

   c) $0 < |z-2| < 2$.

7. (15) Let Ω be a bounded, simply connected domain in $\mathbf{C}$. Suppose that for each $n = 1, 2, \ldots,$ f_n is analytic in Ω and that $\sum_{n=1}^{\infty} f_n(z)$ converges to $S(z)$ uniformly on Ω. Prove:

   a) $S(z)$ is analytic.

   b) $S'(z) = \sum_{n=1}^{\infty} f'_n(z), \quad \forall z \in \Omega.$

8. (10) Show that the series

$$\zeta(z) := \sum_{n=1}^{\infty} n^{-z}$$

   converges absolutely for any z with Re $z > 1$.

9. (15) Let f be an analytic function of a domain A onto a domain B. Let $w \in B$ and let γ be a small circle in A centered at $z_0 \in A$. Suppose that f is one-to-one and f' is nowhere zero. Prove that

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{z f'(z)}{f(z) - w} dz$$

   for w sufficiently close $f(z_0)$.