# 미분기하 학기말고사 (2003년 6월 16일)

1번 - 6번중 택5 (각 5점)

1. Show that a vector $\vec{v} = (v_1, v_2, v_3)$ is tangent to the graph $z = f(x, y), \quad \nabla f \neq 0$, at a point $P = (p_1, p_2, p_3)$ if and only if

$$v_3 = \frac{\partial f}{\partial x}(p_1, p_2)v_1 + \frac{\partial f}{\partial y}(p_1, p_2)v_2.$$

2. Let $\overrightarrow{F} = f_1(x, y)\overrightarrow{U_1} + f_2(x, y)\overrightarrow{U_2}$ be a smooth vector field defined on a domain $D \subset \mathbb{R}^2$ and its smooth boundary $\partial D$. Show that the classical divergence theorem

$$\iint_D \operatorname{div} \overrightarrow{F} dxdy = \int_{\partial D} \overrightarrow{F} \cdot \vec{n} ds$$

is a special case of the Stokes' theorem.

3. Let $M$ be a surface in $\mathbb{R}^3$. Prove:
There exists a nonvanishing 2-form on $M$ if and only if there exists a unit normal vector field on $M$.

4. Show that the cone $z^2 = x^2 + y^2 \quad (z > 0)$ is a ruled surface.

5. Suppose that $\{E_1, E_2, E_3\}$ is a frame field on $\mathbb{R}^3$ and $\omega_{ij} \quad i, j = 1, 2, 3$ are 1-forms defined by

$$\omega_{ij}(v) = \nabla_v E_i \circ E_j$$

for any tangent vector $v$ to $\mathbb{R}^3$ . Show that $\omega_{ji} = -\omega_{ij}$.

6. Let $M$ be a surface and $\sigma$ be a normal section parametrized by arclength. Show that $S(\sigma') \circ \sigma' = \pm k_\sigma, \quad$ where $k_\sigma$ is the curvature of $\sigma$.

7번 - 11번중 택4 (각 8점)

7. Let $\alpha(t) = (a\cos t, \quad a\sin t, \quad bt)$.
a) Reparametrize $\alpha$ by arclenth $s$.
b) Find the Frenet frame, the curvature and the torsion of the curve.

8.Let $M$ be the graph $z = xy - x^3 + 4xy^2$ and let $P$ be the origin.
a) Describe the shape operator $S : T_P M \to T_P M$.
b) Find the principal vectors at $P$.
c) Evaluate the Gaussian curvature and the mean curvature at $P$.

9. Let $X(u, v)$ be a coordinate patch of a surface and $S$ be the shape operator.
Prove:
a) $\nabla_{X_u} X_v = \nabla_{X_v} X_u$.
b) $S(X_u) \circ X_v = S(X_v) \circ X_u$ (i.e.,$S$ is symmetric.)

10. Let $C$ be the circle in $xz$ plane given by
$x = R + r\cos u, \quad z = r\sin u, \quad -\pi < u < \pi$, and let $T$ be the surface of revolution obtained by rotating $C$ around the $z-$axis in the angles $-\pi < v < \pi$. Then the $u$-curves are called meridians and the $v$-curves are called parallels.
a) What are the image curve under the Gauss map $G : T \to \Sigma$ of the meridians and the parallels? Here $\Sigma$ is the unit sphere.
b) Sketch the regions in the parameter rectangle $-\pi < u < \pi, -\pi < v < \pi$ where the Gaussian curvature $K$ is positive, zero and negative, respectively.

11. Consider the saddle surface $M$ parametrized as $X(u, v) = (u, v, uv)$.
a) Find the unit normal vector field $U$ on $M$.
b) Find the Gaussian curvature at $(u, v, uv)$.

총 25+32=57점 x 2.7(가중치) = 154 점 끝