# 다변수해석 학기말고사

2005년 6월 10일 오후 7시 - 9시, 200점 만점

**1** (40) Let $f = (f^1, f^2) : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f^1(x, y, z) = x^2 + y^2 + z^2, \quad f^2(x, y, z) = xe^y + \sin z.$$

Let $P = (1, -1, 0)$.

a) Find $Df(P) = f'(P)$ as a matrix with respect to the standard bases of euclidean spaces.

b) Does there exist a smooth function $z = g(x, y)$ on a neighborhood of $x = 1, y = -1$ such that $f(x, y, g(x, y)) = f(P) = (2, e^{-1})$?

c) Let $\gamma(t) = (1 + at, -1 + bt, ct)$ be a line through $P$ in the direction $(a, b, c)$ and let $\Gamma(t) = (f \circ \gamma)(t)$. Find $\Gamma'(0)$.

d) Let $dA$ be the area element of $\mathbb{R}^2$. Find $f^*dA$.

**2** (15) Consider a 1−form

$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

on $\Omega = \mathbb{R}^2 \backslash \{O\}$.

a) Show that $\omega$ is closed.

b) Let $S \subset \mathbb{R}^2$ be the unit circle centered at the origin. Evaluate $\int_S \omega$.

**3** (30) a) Let $M$ be a compact oriented $k$−dimensional manifold with boundary and $\partial M$ be the boundary of $M$ with the induced orientation. Let $\omega$ be a $(k-1)$−form on $M$. Then state the Stokes' theorem.

b) State the following classical theorems:
Green's theorem,
Divergence theorem,
Stokes' theorem.

c) Using the Stokes' theorem of a) prove the Divergence theorem.

4 (20)Prove that if $A \subset \mathbb{R}^n$ is an open set star-shaped with respect to 0, then every closed $1-$form on $A$ is exact.

5 (15) Show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

6 (20) Let $S \subset \mathbb{R}^3$ be an ellipsoid $x^2 + y^2 + \frac{1}{4}z^2 = 1$. Let $\vec{n}$ be the outward unit normal vector field on $S$. Given a vector field $\vec{F} = (x^2, e^y, z)$ evaluate

$$\iint_S (\mathrm{curl} F \bullet \vec{n}) dA.$$

7 (20) Let $\Omega \subset \mathbb{R}^2$ be a compact manifold with boundary and $\vec{n}$ be the outward unit normal vector field on $\partial\Omega$. Prove that if $u$ is a smooth function on a neighborhood of $\Omega$ then

$$\int_{\partial\Omega} (\nabla u \bullet \vec{n}) ds = \iint_{\Omega} (u_{xx} + u_{yy}) dA.$$

8 (20) Suppose $f(z) = u(x, y) + iv(x, y)$ be a smooth complex valued function that satisfies $u_x = v_y$ and $u_y = -v_x$, where $z = x + iy$ and $dz = dx + idy$. Let $C$ be a smooth closed curve in $\mathbb{R}^2$. Show that $\int_C f(z)dz = 0$.

9 (20) Consider the 2-form $\omega$ defined on $\mathbb{R}^3 \backslash \{0\}$ by

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Evaluate the integral $\int_S \omega$, where $S$ is a sphere of radius 1 centered at $(1, 1, 1)$.