다변수해석 중간고사

2005년 4월 20일 오후 7시 - 9시, 150점 만점

1 (20) State the theorems precisely:

a) Inverse function theorem.
b) Implicit function theorem.
c) Fubini's theorem.
d) Partition of unity theorem.

2 (15) In each of the following cases the given set is a subset of $\mathbb{R}$. Describe the boundary, interior, and the closure of the set as subsets of $\mathbb{R}$.
a) A is the set of rational numbers in $[0,1]$. Describe ∂A, A° and $\bar{A}$.
b) $B = \{1,2,3\}$.
c) $C = \{\frac{1}{n} \quad | \quad n : \text{positive integers}\}$.

3 (10) Suppose $u = f(x,y)$, $x = s^2 - t^2$, $y = 2st$. Assuming f is of class C^2, find $\frac{\partial^2 u}{\partial s \partial t}$ in terms of s, t and the derivatives of f.

4 (15) Let $f(x,y) = x^2 + 5xy$. At the point $(-2,1)$
a) find the directional derivative of f in the direction of the unit vector $\vec{v}/\|\vec{v}\|$, where $\vec{v} = (12,5)$.
b) what is the largest of the directional derivatives of f at this point, and in what direction does it occur?

5 (15)

$$\begin{cases} xy^2 + xzu + yv^2 & = 3 \\ u^3yz + 2xv - u^2v^2 & = 2. \end{cases}$$

a) Is it possible to solve for u and v in terms of x, y and z on a neighborhood of $(x,y,z) = (1,1,1)$, $(u,v) = (1,1)$.
b) Express du and dv as linear combinations of dx, dy, dz with real coefficients at the point $(x,y,z,u,v) = (1,1,1,1,1)$.

6 (30) Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (2x + y^2, \quad y + y^3)$.
a) Let $\gamma(t) = (\sin t, te^t)$ and let $\Gamma(t) = (f \circ \gamma)(t)$. Find $\Gamma'(0)$.
b) Show that f is one-to-one on the rectangle $A = [0,1] \times [0,1]$.
c) Find the area of $f(A)$.

7번 이하는 증명문제, 7-8 중 택1, 9-10 중 택일, 11-12중 택일

7 (15) If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $a \in \mathbb{R}^m$, then f is continuous at a.

8 A bounded function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous at $a \in \mathbb{R}^m$ if and only if the oscillation of f at a is zero.

9 (10) Let $A \subset \mathbb{R}^n$ be a closed rectangle and $g : A \to \mathbb{R}$ is a non-negative integrable function such that $\int_A g = 0$. Show that $g = 0, a.e.$

10 If f is a C^2 real-valued function defined on an open $\Omega \in \mathbb{R}^2$ then the partial derivatives of f in mixed orders are equal, i.e., $D_x D_y f = D_y D_x f$ at every point of Ω.

11 (20) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function.
a) If $S \subset \mathbb{R}^n$ is an open convex set and $|\nabla f(x)| \leq M$ for every $x \in S$, then $|f(b) - f(a)| \leq M|b - a|$ for all $a, b \in S$.
b) If $S \subset \mathbb{R}^n$ is an open convex set and $\nabla f(x) = 0$ for all $x \in S$, then f is constant on S.

12 a) Suppose $Z \subset \mathbb{R}^2$ has content zero. If $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f = 0$,
b) Suppose that f is integrable on the set $S \subset \mathbb{R}^2$. If $g(x) = f(x)$ except for x in a set of content zero, then g is integrable on S and $\int_S g = \int_S f$.
c) Suppose that f is integrable on S and on T and that $S \cap T$ has content zero. Then f is integrable on $S \cup T$ and

$$\int_{S \cup T} f = \int_S f + \int_T f.$$