# 미분다양체론 기말고사

2005년 6월 14일, 19시 200점 만점

1 (25)For each of the following tell whether true or false:

a) Every smooth vector field on $\mathbb{R}^1$ is complete.

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 2x^2 + 4xy + y^4$. Then $f^{-1}(-1)$ is an embedded submanifold of $\mathbb{R}^2$.

c) Möbius band is not orientable.

d) $n$-dimensional real projective space $P^n$ is not orientable if $n$ is even.

e) Dimension of $\Lambda^k(M^n)$ is $\begin{pmatrix} n \\ k \end{pmatrix}$.

2 (10) Discuss the de Rham cohomology vector spaces $H^p_{deR}(S^2)$ for $p = 0, 1, 2$.

3 (20) In $\mathbb{R}^2$ we consider vector fields $X, Y$ and a $1-$form $\omega$ given by $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and $\omega = ydx - xdy$.

a) Compute $L_X Y$.

b) Compute $L_X \omega$

4 (25)Let $G = SL(2, \mathbb{R})$ be the set of all $2 \times 2$ real matrices with determinant 1.

a) Show that $G$ is a closed subgroup of $GL(2, \mathbb{R})$. (This implies $G$ is a Lie group.)

b) Show that if $\{e^{tA} : t \in \mathbb{R}\}$ is a $1-$parameter subgroup of $G$ then trace of $A$ is zero.

5 (20)a)Suppose that a $3 \times 3$ real matrix $A$ preserves inner product of any two vectors in $\mathbb{R}^3$, that is, $< AX \quad AY >=< X \quad Y >$ for all 3-dimensional column vectors $X$ and $Y$. Show that $A$ satisfies $AA^t = I$. We shall denote by $O(3)$ the set of all such matrices.

b) Find a basis of $o(3)$ the Lie algebra of $O(3)$.

6 (20)Prove that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is not $e^A$ for any $A \in gl(2, \mathbb{R})$.

7 (20) Let $G$ be a Lie group and let $\mathcal{G}$ be its Lie algebra. Suppose that $\{X_1, \ldots, X_d\}$ is a basis of a subalgebra of $\mathcal{G}$. Let $H$ be a maximal connected integral manifold of $\{X_1, \ldots, X_d\}$ through the identity element $e$ of $G$. Show that $H$ is a Lie subgroup of $G$.

8 (20)a) Let $\omega = \frac{-ydx+xdy}{x^2+y^2}$ and $\alpha$ be a $1-$form on $S^1$. Then there exists a constant $c \in \mathbb{R}$ such that $\alpha - c\omega$ is exact.

b) What do you conclude from a) on $H^1_{deR}$?

9 (20)Let $\omega = \frac{xdy\wedge dz+ydz\wedge dx+zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$.

a) Show that $\omega$ is closed.

b) Evaluate $\int_M \omega$, where $M$ is a sphere centered at (2,0,0) of radius 1.

10 (20)Suppose $M$ is a compact oriented connected $n$-manifold with no boundary and $\theta$ is an $(n-1)-$ form on $M$. Show that $d\theta$ is zero at some point of $M$.