미분다양체론 중간고사

2005년 4월 22일, 19시 150점 만점

1 (15) Let $x=(x_1,x_2)$ be a local coordinate system centered at $m \in M^2$ and $y=(y_1,y_2,y_3)$ be a local coordinate systems of N^3. Let $\Psi : M \to N$ be a smooth mapping whose coordinate expression $y \circ \Psi \circ x^{-1}$ is given by

$$(x_1,x_2) \mapsto (\sin x_1, x_2 + (x_1)^3, x_1 + x_2).$$

Let $V \in T_mM$ be given by $V = 2\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$. Find $d\Psi_m(V)$.

2 (15) Consider the function $f(x,y) = xy$ defined on $\mathbb{R}^2 = \{(x,y)\}$. In each of the following tell whether the level set is a smooth submanifold. Justify your answer.

a) $f^{-1}(0)$.

b) $f^{-1}(1)$.

3 (20) In $\mathbb{R}^2 = \{(x,y)\}$ let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $\omega = ydx - xdy$. Find

a) $d\omega$

b) $L_X\omega$

c) $dL_X\omega$

d) $L_Xd\omega$.

4 (15) Let M be a compact manifold of dimension n and $f : M \to \mathbb{R}$ be a smooth function. Show that there are at least two critical points, i.e., the points $P \in M$ where $df(P) = 0$.

5-6 중 택일 (25점)

5

Let X_1 and X_2 be vector fields on $\mathbb{R}^4$ given by

$$X_1 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_4}.$$

a) Is there a 2-dimensional submanifold M^2 of $\mathbb{R}^4$ such that for each $P \in M^2$, $X_1(P), X_2(P) \in T_PM$?

b)Is there a nonconstant function f in a neighborhood of $O \in \mathbb{R}^4$ such that $X_1f = 0$ and $X_2f = 0$?

6

On $\mathbb{R}^3 = \{(x,y,z)\}$ consider two vector fields $X = \frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y} + \{1 + e^x(x^2 + y - z)\}\frac{\partial}{\partial z}$. Does there exist a 2-dimensional integral manifold of the distribution spanned by X and Y ? Explain why or why not.

다음 문제중 3개 선택 (각 20점)

7 Let $X_1, \cdots, X_c, \quad 2 \leq c \leq n$, be smooth vector fields on M^n. Let $x = (x_1, \cdots, x_n)$ be coordinate system such that $X_1 = \frac{\partial}{\partial x_1}$.

a) Let $Y_j = X_j - (X_j x_1) X_1, \quad j = 2, \cdots, c$. Show that $Y_j, \quad j = 2, \cdots, c$, is tangent to the submanifold $S = \{x_1 = 0\}$.

b) If $X_1, \cdots, X_c$ is involutive then $Y_2, \cdots, Y_c$ is involutive on S.

8 Completeness of vector fields: Tell yes or no. Justify your answers (give proof for yes and counter-example for no).

a) Is every smooth vector field on the real line complete?

b) Is every smooth vector field on a compact manifold complete?

9 $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4\}$ satisfying

$$\operatorname{rank} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = 1$$

is a three dimensional submanifold of $\mathbb{R}^4$.*Prove* :

10

Define a 2-form on $\mathbb{R}^{2n}$ by

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

This is called the standard symplectic form on $\mathbb{R}^{2n}$. Compute $\omega^n = \omega \wedge \cdots \wedge \omega$, n-times wedge product.