Complex analysis Final Exam

December 12th, 2008

1 (15 pts = 7+8) Find all the values of

a) 6^{th} root of -8.
b) i^i.

2 (20 pts = 7+7+6) Mappings:

a) Find the image of $\{z : |\text{ Im } z| < \frac{\pi}{3}\}$ under the exponential function $w = e^z$.
b) Find the image of the line $y = x$ under the mapping $w = \frac{z-1}{z+1}$.
c) If $f(z) = u(x, y) + iv(x, y)$ is analytic and $f'(z) \neq 0$ at any point of its domain, then the level curves of u intersect perpendicularly with the level curves of v.

3 (20 pts = 5+10+5) Laurent series:

Let $f(z)$ be analytic on a region that contains a closed annulus $r_1 \leq |z - a| \leq r_2$. Then
a) for any z in the interior of the annulus we have $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r_2} \frac{f(\zeta)d\zeta}{\zeta - z}, \qquad f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta - a| = r_1} \frac{f(\zeta)d\zeta}{\zeta - z}.$$

b) $f_2(z) = \sum_{k=1}^{N} \frac{b_k}{(z-a)^k} + \sigma_N(z), \quad$ where

$$b_k = \frac{1}{2\pi i} \int_{|\zeta - a| = r_1} (\zeta - a)^{k-1} f(\zeta)d\zeta, \qquad \sigma_N(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r_1} \frac{(\zeta - a)^N f(\zeta)d\zeta}{(z - a)^N (z - \zeta)}.$$

c) Let $r > r_1$. Then $\sigma_N(z) \to 0$ uniformly on the closed set $|z - a| \geq r > r_1$.

4 (20 pts= 10+10) Find the residue at each pole:

a) $\dfrac{\cosh(\pi z)}{z^2(z^2 + 1)}$
b) $\dfrac{\text{Log } z}{(z^2 + 1)^2}$

5 (10) Prove that $f(z) := 3z^{100} - e^z$ has 100 roots in the unit disk $|z| < 1$ and all of them are simple zeros.

6 (10) Let $f(z)$ be analytic on a region Ω containing a circle $|z| = 2$ and its interior. If $f(z)$ has a zero of order m at $\frac{\pi i}{2}$ and no other zeros in Ω. Evaluate $\displaystyle\int_{|z|=2} \frac{zf'(z)}{f(z)} dz$.

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7 (25 pts = 10+15)Laplace transform:

a) Let $F(s) = \dfrac{4s+6}{s^2(s+2)(s-1)}$. Find $f(t) = \mathcal{L}^{-1}(F(s))$ by contour integral.

b) By the Laplace transform method solve the following initial value problem of the system of differential equations for $x(t)$ and $y(t)$:

$$\begin{aligned} x'' - 2x' + 3y' + 2y &= 4, \\ 2y' - x' + 3y &= 0, \\ x(0) = x'(0) = y(0) &= 0. \end{aligned}$$

8 (20 pts = 10+10) Linear Fractional Tranformations:

a) For a linear fractional transformation T, find $T(2)$ if

$$T(1) = 1 + i, \quad T(0) = -i, \quad T(-i) = 0.$$

b) A fixed point of a transformation $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$. Show that every linear fractional transformation, with the exception of the identity transformation $f(z) = z$, has at most two fixed points in the extended plane.

9 (25 pts = 15 for a, b, c + 10 for d.)

a) Show that $\pi \cot \pi z$ has residue $+1$ at each of the poles $z = 0, \pm 1, \pm 2, \ldots$.

b) For each positive integer N let C_N be the square with vertices at $\pm(N+\frac{1}{2}) \pm i(N+\frac{1}{2})$. Then $\pi \cot \pi z$ is bounded on C_N, the bound being independent of N.

c) Consider $\displaystyle\int_{C_N} \frac{\pi \cot \pi z}{z^2} dz,$ show that the integral tends to zero as $N \to \infty$.

d) Evaluate $\displaystyle\sum_{1}^{\infty} \frac{1}{n^2}$.

10 (15 pts) By analogy with the above, integrate $\dfrac{\pi \csc \pi z}{z^2}$ around C_N and evaluate $\displaystyle\sum_{0}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

End of problem set. 180 points total.