Complex analysis (Undergrad) Midterm Exam

October 17th, 2008

1 (5+5+5=15 pts) State the theorem with precise hypotheses and conclusions:
a) Cauchy-Goursat theorem.
b) Morera's theorem.
c) Liouville's theorem.

2 (5+5+5=15) Let $u(x,y) = (e^y + e^{-y})\cos x$.
a) Show that u is harmonic in $\mathbb{C}$.
b) Find a conjugate harmonic function.
c) Evaluate

$$\iint_{|z|<\pi} u(x,y)dxdy.$$

3 (5+5+10=20) Evaluate the integral $\displaystyle\int_C \frac{\sin(\pi z)dz}{(z+\frac{3}{2})(z-4)^2}$, where
a) C is the circle $|z| = 1$.
b) C is the circle $|z| = 3$.
c) C is the circle $|z| = 5$.

4 (10+10=20) Prove:
a) Suppose that f is entire and $|f(z)| \leq M|z|^n$ whenever $|z| \geq R$, for some positive numbers M and R. Then $f^{(n)}$ is constant.
b) If $u(x,y)$ is positive-valued and harmonic at all points in $\mathbb{R}^2$ what can you conclude about u? Justify your answer.

5 (10) Evaluate $\displaystyle\int_{|z-i|=2} \frac{\mathrm{Log}(z+2)}{z-2i}dz.$

6 (10+10=20) Let D be the unit disk $|z| < 1$ and $f : \bar{D} \to \bar{D}$ be analytic. Prove
a) If $|f(z)|$ is constant then f is constant.
b) Suppose that $|f(e^{i\theta})| = 1$ for all $\theta \in [0, 2\pi]$ and f has no zeros in D. Show that f is constant.

7 (10+10=20) Define a complex-valued function $g(z)$ defined on $|z| \neq 1$ by

$$g(z) = \int_{|\zeta|=1} \frac{e^{2\zeta}}{(\zeta - z)^2}d\zeta.$$

a) Find $g'(z)$.
b) Show that g is discontinuous at $z = 1$ by finding $\lim_{t\to 1^-} g(t)$ and $\lim_{t\to 1^+} g(t)$, where t is real.

End of problem set. 120 points total.