Complex analysis exercises on Real Integrals

November 21st, 2008

1 (15pts $\times$ 4 = 60) Evaluate the integrals. Show the convergence.

a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$.

c) $\displaystyle\int_{0}^{\infty} \frac{x^{1/3}}{1+x^2} dx$.

d) $\displaystyle\int_{0}^{\infty} \frac{\sin ax}{x} dx \qquad (a > 0)$.

2 (10 pts) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

3 (10 pts $\times$ 3 = 30) Follow the steps below to evalutae the "Fresnel integrals":

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

a) By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R, \quad 0 \le \theta \le \pi/4$ and appealing to the Cauchy-Goursat theorem, show that

$$\int_0^R \cos(x^2)dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr - \mathrm{Re}\int_{C_R} e^{iz^2}dz$$

and

$$\int_0^R \sin(x^2)dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr - \mathrm{Im}\int_{C_R} e^{iz^2}dz,$$

where C_R is the arc $z = Re^{i\theta}, \quad 0 \le \theta \le \pi/4$.

b) Show that the value of the integral along the arc C_R in part (a) tends to zero as R tends to infinity by obtaining the inequality

$$|\int_{C_R} e^{iz^2}dz| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}d\phi$$

and then referring to the Jordan's inequality.

c) Use the results in (a) and (b), together with the known integration formula $\displaystyle\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, to complete the Fresnel integrals.

End of problem set. 100 points total.