Multi-variable calculus Final Exam

June 12, 2008

1 Stokes' theorem: (10+10+10=30 pts)

a) State and prove the Stokes' theorem (differential form version).

b) Green's theorem: State the Green's theorem and prove it by using a).

c) Divergence theorem in $\mathbb{R}^4$: Let $\Omega \subset \mathbb{R}^4$ be a connected domain with C^1 boundary. State the divergence theorem for vector fields defined on a neighborhood of $\overline{\Omega}$ and prove the theorem by using a).

2 State the following theorems: (5+5=10 pts)
a) Inverse function theorem
b) Poincaré lemma

3 (10 pts) Let $f(x, y, z) = xyz + x^2 - 4z^2 + 5y - 1$ and B be the unit ball $x^2 + y^2 + z^2 \leq 1$. Find the flux of the gradient field ∇f across the boundary of the ball.

4 (10+10+10 = 30 pts) In $\mathbb{R}^3$ let $\vec{r} = (x, y, z)$ be the position vector and $r := \|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$.

a) Find $\nabla(\frac{1}{r})$ and $div\nabla(\frac{1}{r})$.

b) Let $\Omega \subset \mathbb{R}^3$ be a domain with C^1 boundary that contains the origin in its interior. Show that

$$\int_{\partial\Omega} \nabla(-\frac{1}{r}) \bullet \vec{n} dS = 4\pi.$$

c) Let ρ be a radial function on $\mathbb{R}^3$ defined by

$$\rho(r) = \begin{cases} 1 - r & \text{if } 0 \leq r \leq 1 \\ 0 & \text{if } r \geq 1 \end{cases} .$$

Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary containing the ball $r \leq 2$. Define a function ϕ on $\mathbb{R}^3$ by

$$\phi(x) = \int_\Omega \frac{\rho(y) dV(y)}{4\pi \|x - y\|}.$$

Evaluate

$$\int_{\partial\Omega} \frac{\partial \phi}{\partial n} dS.$$

5 (10 pts) Evaluate $\iint_S (\nabla \times \vec{F}) \bullet \vec{n} dS$, where $\vec{F} = (x - z, x^3 + yz, -3xy^2)$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy plane.

6 (15 pts) Let $\Omega \subset \mathbb{R}^2$ be the region bounded by the coordinate axes and the line $x+y=1$. Use the substitution $u = x-y$, $v = x+y$ to evaluate

$$\int_{\Omega} e^{\frac{x-y}{x+y}} dxdy.$$

7 (9+8+8=25) volume of a solid torus in $\mathbb{R}^3$ obtained by revolving the circle $(y-a)^2 + z^2 \leq b^2$, $a > b$, in the yz-plane, about the z-axis:

a) Consider the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(u,v,w) = (x,y,z)$, where

$$x = (a + w\cos v)\cos u$$
$$y = (a + w\cos v)\sin u$$
$$z = w \sin v.$$

Compute $T^*(dx \wedge dy \wedge dz)$. (Express in terms of u, v, w.)

b) Let $Q := \{(u,v,w) \in \mathbb{R}^3 : u, v \in [0, 2\pi], w \in [0,b]\}$. Evaluate

$$\int_Q T^*(dx \wedge dy \wedge dz).$$

c) Sketch the curves $T(u,0,b)$, $T(0,v,b)$ on the torus.

8 (10+10+10=30) 2-form on $\mathbb{R}^3 \setminus O$ which is closed but not exact:

a) Let S be the unit sphere in $\mathbb{R}^3$ and ω be an exact 2-form. Show that $\iint_S \omega = 0$.

b) Consider the 2-form ω defined on $\mathbb{R}^3 \setminus O$ by

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

Evaluate $\iint_S \omega$.

c) Show that ω is closed but not exact.

End of problem set, total 160 pts.