다변수해석학 중간고사

2008. 4.19

문제 1-9 제출, 문제 10-17 화요일 4/22 수업시간에 제출

1 State the theorem. (5+5+5=15 pts)
a) Implicit function theorem
b) Fubini theorem
c) Partition of unity

2 Evaluate: (5+10+15=30 pts)
a) $\int_{-\infty}^{\infty} e^{-x^2} dx$.
b) For $x > 0$, find

$$\frac{d}{dx} \int_{x^2}^{x^3} \frac{1}{y} e^{xy^2} dy.$$

c) Let R be the region in the first quadrant that is bounded by the hyperbolas $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$. Express the integral $\int_R (x^2 + y^2) dxdy$ in terms of the new variables $u = xy$ and $v = x^2 - y^2$. Then evaluate the integral.

3 (5+15=20 pts) Let $\gamma(t) = (\cosh t, \sinh t - 1)$ be a curve in $\mathbb{R}^2$. Recall $\cosh t = \frac{e^t + e^{-t}}{2}$.
a) Find $\gamma'(t)$.
b) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $f(x, y) = (ye^x, \sin(x+y), y^2+1)$ and let $\Gamma(t) = (f \circ \gamma)(t)$. Find $\Gamma'(0)$.

4 (10+10=20 pts) a) Let A be an $n \times n$ matrix with $\det A \neq 0$. Show that there exist positive numbers m and M such that $m\|v\| \leq \|Av\| \leq M\|v\|$ for any non-zero vector $v = (v_1, \ldots, v_n)^t \in \mathbb{R}^n$.

b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and that $Df(a) \neq 0$. Show that f is one-to-one on a neighborhood of a.

5 (12 pts) Suppose that $f(x, y, x)$ is C^∞ and $f(x, 0, 0) = 0$. Then there exist functions g and h such that $f(x, y, z) = yg(x, y, z) + zh(x, y, z)$: Prove.

6 (10 pts) If $f(x, y)$ is a C^2 real-valued function, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Recall $\frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$.

7 (13 pts) Let $A \subset \mathbb{R}^n$ be a rectangle and $f : A \to \mathbb{R}$ be a non-negative function. If $\int_A f = 0$, show that $\{x : f(x) \neq 0\}$ has measure 0.

8 (10+5=15 pts) Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

a) Find the oscillation of $h(x) := \cos x + f(x) \sin x$.

b) Discuss the continuity of $h(x)$.

9 (15 pts) Find the points of the ellipsoid $x^2+2y^2+3z^2=1$ which are closest to and farthest from the plane $x+y+z=10$.

class exam total 150 pts

.... take home ....

10 Let $f(x,y,z) = x\sin z - z\sin y$.
a) Find the Taylor expansion up to degree 5.
b) Find the hessian of f at $(0,0,0)$. Recall the hessian the symmetric matrix of the second derivatives.
c) Tell whether each of the following points is a critical point. If so, classify the critical point: $(0,0,0)$, $(-1,\pi/2,0)$.

11 For the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y\sqrt{x^2+y^2}}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

a) Show that f is continuous at $(0,0)$.
b) Show that $D_1f(0,0) = 0$, $D_2f(0,0) = 0$.
c) Is f is differentiable at $(0,0)$? Prove or disprove.

12 Let $a(x), b(x), g(x,y)$ be C^∞ functions. Show that

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,y)dy = g(x,b(x))b'(x) - g(x,a(x))a'(x) + \int_{a(x)}^{b(x)} D_1g(x,y)dy.$$

13 Assuming the implicit function theorem prove the inverse function theorem.

14 Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function such that $\int_{[0,1]\times[0,1]} fg = 0$ for any $g : [0,1]^2 \to \mathbb{R}$. Show that f is constantly zero.

15 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $V \in \mathbb{R}^n$. Define $D_Vf(a) = \lim_{t\to 0} \frac{f(a+tV)-f(a)}{t}$ if the limit exists. Show that if f is differentiable at a then $D_Vf(a) = Df(a)V$.

16 Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Show that $\{x | f \text{ is discontinuous at } x\}$ has measure 0.

17 Find the volume of the unit ball $\|x\| \leq 1$ in $\mathbb{R}^n$, for $n = 1,2,3,4$.

end of problem set