Complex analysis (Grad) Final Exam

December 15th, 2008

1 Taylor and Laurent series:

*a)(10 pts) If f is holomorphic in an open neighborhood of the closed disk $|z-a| \leq R$ and z is a point in the interior of the disk, then we have

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + f_{n+1}(z)(z-a)^{n+1},$$

where

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=R} \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}(\zeta-z)}.$$

*b) (10 pts) Let f be the same as in a). Then its Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$ converges uniformly on compact subsets of the disk $|z-a| < R$.

c) Let f be holomorphic in a region that contains the annulus $r_1 < |z-a| < r_2$. Then for any point z in the interior of the annulus we have $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r_2} \frac{f(\zeta)d\zeta}{\zeta - z}, \quad \text{and } f_2(z) = -\frac{1}{2\pi i}\int_{|\zeta-a|=r_1} \frac{f(\zeta)d\zeta}{\zeta - z}.$$

d) Express f_1 and f_2 as power series in $(z-a)$ and $\frac{1}{z-a}$, respectively, and discuss the convergence.

2 Partial fractions:

a) State and prove the Mittag-Leffler theorem.

b) Prove that $\displaystyle\frac{\pi^2}{\sin^2(\pi z)} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}$.

c) By differentiating b) show that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

d) Show that

$$\begin{aligned}\frac{\pi}{\sin \pi z} &= \lim_{m\to\infty} \sum_{-m}^{m} (-1)^n \frac{1}{z-n} \\ &= \frac{1}{z} + \sum_{1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2}.\end{aligned}$$

e) Comparing coefficients in the Laurent developments of $\cot \pi z$ and of its expresson as a sum of partial fractions, evaluate $\displaystyle\sum_{1}^{\infty} \frac{1}{n^2}$.

*f)(10 pts) Find the partial fraction development of $\displaystyle\frac{1}{\cos \pi z}$ and evaluate $1 - 1/3 + 1/5 - 1/7 \cdots$.

3 a) Show that $\sin \pi z = \pi z \prod_{n\neq 0}(1-\frac{z}{n})e^{z/n}$.

b) Prove that the right hand side of a) converges absolutely for any z. Therefore, we can change the order of multiplication: multiplying the terms $\pm n$ as a pair, then we have

$$\sin \pi z = \pi z \prod_1^\infty \left(1-\frac{z^2}{n^2}\right).$$

4 Gamma function: a) We define an entire function with simple zeros at negative integers: $G(z) = \prod_1^\infty (1+\frac{z}{n})e^{-z/n}$. Then

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi}.$$

b) $G(z-1) = e^\gamma z G(z)$ defines the Euler constant

$$\gamma = \lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n).$$

Let $H(z) := G(z)e^{\gamma z}$ and we define

$$\begin{aligned}\Gamma(z) :=& 1/[zH(z)]\\ =&\frac{e^{-\gamma z}}{z}\prod_{n=1}^\infty\left(1+\frac{z}{n}\right)^{-1}e^{z/n}.\end{aligned}$$

Show that

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^\infty \frac{1}{(z+n)^2}.$$

c) By using the Stirling's formula prove $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$.

5 a) Show that $\Gamma(z+1) = z\Gamma(z)$.

b) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$.

c) Evaluate $\Gamma(1/2)$, $\Gamma(1)$, $\Gamma(2)$.

d) Find the residue of $\Gamma(z)$ at 0, and at -1.

6 * a)(10 pts) Define the genus and the order for an entire function.

*b)(10 pts) Find the genus and the order of $\sin(z^2)$.

7 Riemann Zeta Function:

a) Show that the series $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, $s = \sigma + it$, converges absolutely and uniformly on compact subsets of the half plane $\sigma > 1$, and therefore, $\zeta(s)$ is holomorphic in $\sigma > 1$.

b) For $\sigma > 1$ show that

$$\zeta(s) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n{}^s}\right)^{-1},$$

where $p_1, p_2, \cdots$ are the prime numbers.

c) Show that $\displaystyle\int_C \frac{(-z)^{s-1}}{e^z - 1} dz$ is an entire function of s, where C is the infinite path beginning and ending near the positive real axis and turns around the origin along a positively oriented circle of a small radius $\rho < 2\pi$. Here $(-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi < \text{Im} \log(-z) < \pi$.
(Hint: For any real number $A > 0$ let C_A be the finite path $\{z \in C : \text{ Re } z \leq A\}$. Then the integral over C_A is an entire function of s and the integral over $C \setminus C_A$ converges uniformly to zero on $|s| \leq R$ as $A \to \infty$.)

*d)(10 pts) Recalling $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$, show that for $\sigma > 1$,

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz.$$

*e)(10 pts) Show that

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s).$$

f) Find the residues at the poles of $\zeta(s)$ and the zeros of $\zeta(s)$ in $\sigma > 1$ and in $\sigma < 0$.

g) Discuss the Riemann hypothesis.

8 Normal families:

a) State and prove the Arzela-Ascoli theorem.

b) Let $\mathcal{F}$ be a family of holomorphic functions of a domain Ω into a bounded domain $\tilde{\Omega}$. Show that $\mathcal{F}$ is a normal family.

c) Show that $\{f' : f \in \mathcal{F}\}$, where $\mathcal{F}$ is as in b), is a normal family.

*d)(10 pts) Let Ω be a simply connected domain and f be a nowhere zero holomorphic function on Ω. Show that there are holomorphic functions ϕ and g on Ω such that $e^{\phi} = f$, and $g^2 = f$, respectively.

*e)(10 pts) Let Ω be an unbounded simply connected domain which is not the whole plane. Let $z_0 \in \Omega$. Construct a univalent (one-to-one holomorphic) function f from Ω into the open unit disk U such that $f(z_0) = 0$ and $f'(z_0) > 0$. Justify your answer.

*f)(10 pts) Let H be the upper half plane $y > 0$ and U be the open unit disk. Find the Riemann map $f : (H, i) \to (U, 0)$, that is, a one-to-one, onto, holomorphic map $f : H \to U$ such that $f(i) = 0, f'(i) > 0$.

g) Assuming all the above results prove the Riemann mapping theorem.

9 *(15 pts) Find a meromorphic function with the poles of the following singular part: Express as an infinite series that converges uniformly on compact sets after omitting finite number of terms.

*a) $\dfrac{1}{z - n\log n}$, $n = 1, 2, 3 \ldots$

*b) $\left(\dfrac{1}{z - \sqrt{n}}\right)^2$, $n = 1, 2, 3 \ldots.$

10 *(20 pts = 7+7+6) Find an entire function with

*a) simple zeros at $n\log n$, $\quad n = 1, 2, 3, \ldots,$ and no other zeros

* b) double zeros at $0, 1, 2, 3, \ldots,$ and no other zeros

*c) simple zeros at n^2, $\quad n = 0, 1, 2, \ldots,$ and no other zeros.

11 *(15 pts) Let $f(z)$ be a holomorphic function in a region that contains the closed unit disk $|z| \leq 1$. Suppose that f has zeros at a_j $\,, j = 1, \ldots, N$, in the interior to the unit circle $|z| = 1$ and no other zeros.

*a) Construct a holomorphic function $F(z)$ that has no zeros in $|z| \leq 1$ and such that $|F(z)| = |f(z)|$ on the circle $|z| = 1$.

*b) Express $|f(0)|$ in terms of the boundary values $f(e^{i\theta})$ and a_j's.

12 *(15 pts) Using the probability integral

$$\int_0^\infty e^{-t^2} dt = \frac{1}{2}\int_0^\infty e^{-x} x^{-1/2} dx = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi},$$

evaluate the Fresnel integrals $\displaystyle\int_0^\infty \sin(x^2)dx$, and $\displaystyle\int_0^\infty \cos(x^2)dx$.

13 *(15 pts) For a positive integer N let C_N be the the square with vertices at $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$. By evaluating the integral $\int_{C_N} \frac{\pi \csc \pi z}{z^2} dz$ and by taking the limit $N \to \infty$, evaluate $\displaystyle\sum_1^\infty \frac{(-1)^{n+1}}{n^2}$.

End of problem set. Total 180 points.