Complex analysis (Grad) Midterm Exam

October 21, 2008

1 (10+10=20 pts) Let D be the open unit disk $|z| < 1$.

a) Suppose that $\phi : \bar{D} \to \bar{D}$ is holomorphic, $|\phi(z)| = 1$ if $|z| = 1$, and that ϕ has no zero in D. What is your conclusion on ϕ ? Justify your answer.

b) Suppose that $f : \bar{D} \to \bar{D}$ is holomorphic, $|f(z)| = 1$ if $|z| = 1$. If f has zero of order 2 at $a \in D$ and no other zeros. What do you conclude on f ? Justify your answer.

2 (10) Let $f(z) = 2z^2 + z^3$. Prove that there exist open neighborhoods U and V of the origin such that f maps U onto V in two-to-one manner (two-to-one except for $f(0) = 0$).

3 (10+5=15) Reflection principle:

Let $\Omega^+ := \{(x, y) : y > x^2, \quad |z| < \epsilon\}$, and $\sigma := \{(x, y) : y = x^2, \quad |z| < \epsilon\}$, where $z = x + iy$ and $\epsilon > 0$ is a sufficiently small constant.

a) For a point $z \in \Omega^+$ find the reflection point z^*.

b) State the reflection principle for holomorphic functions on Ω^+.

4 (10+10=20) Poisson kernel: Let D be the open unit disk $|z| < 1$.

a) Let f be holomorphic in $\bar{D}$ and for any $z \in D$ let z^* be the reflection point of z with respect to the boundary of D. Show that

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) f(\zeta) d\zeta.$$

b) By using a) prove the Poisson integral formula: If u is harmonic in $\bar{D}$ then for any $z = re^{i\phi} \in D$

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) P(r, \phi - \theta) d\theta,$$

where $P(r, t) = \frac{1 - r^2}{1 - 2r \cos t + r^2}$.

5 (10+5+10=25) Let $u(r, \theta) = \log r$.

a) Express $*du$ and $d * du$ in terms of x and y.

b) Show that u is harmonic.

c) Let $\Omega := \mathbb{C} \setminus \{0, 2\}$. Find $\displaystyle\int_\gamma *du$ for each cycle γ of the homology basis of Ω. Discuss the existence of harmonic conjugates in Ω.

6 (10+10+10=30) Harmonic functions;

a) If u and u^2 are both harmonic in a domain $\Omega \subset \mathbb{C}$, what do you conclude on u? Justify your answer.

b) If $u_1, u_2, \cdots$, are harmonic in Ω and $\{u_k\}$ converges to u_0 uniformly on each compact subset of Ω, then prove u_0 is harmonic.

c) If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

End of problem set. Total 120 points.