복소함수론 기말 고사 , 1999년 6월 16일

1. (15 점) Let C denote the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$, where C described in the positive sence. Evaluate each of these integrals:

a) $\int_C \frac{z^2}{z-3} dz$

b) $\int_C \frac{\cos z}{z(z^2+8)} dz$

2.(15 점) Find the value of the integral of $f(z)$ around the circle $|z-i| = 2$ in the positive sense when

a) $f(z) = \frac{1}{z^2+4}$

b) $f(z) = \frac{1}{(z^2+4)^2}$

3. (15 점) Suppose that $f(z)$ is an analytic $\forall z \in \mathbb{C}$ and that the real part of $f(z)$ is bounded above, i.e., there is a constant M such that $\text{Re} f(z) \leq M$. Show that $f(z)$ is constant.

4.(30 점) a) Find the Taylor series expansion of $\frac{1}{z^2}$ at $z = 2$. What is the radius of convergence?

b) Find the Laurent series expansions of $f(z) = \frac{1}{z^2(1-z)}$ at $z = 1$ in each region of convergence .

5.(20 점) Find the residues at each singular point.

a) $(\frac{z}{2z+1})^3$

b) $z^3 \exp(\frac{1}{z})$

c) $\tan z$

6.(15점) Let $P(z)$ and $Q(z)$ denote polynomials of degree n and m, respectively, where $m - n \geq 2$. Prove (or explain why): If all of the zeros of the polynomial $Q(z)$ are interior to a simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)} dz = 0.$$

7.(20점) Show that if $a > 0$ then

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \pi e^{-a}.$$

8.(25점) Evaluate $\int_0^\infty \frac{\sin x}{x} dx$ using the residue theorem.

9.(10점) If $a > e$, show that $e^z = az^n$ has n roots in $|z| < 1$.

10. (15점) Let γ be a positively oriented simple closed curve. Suppose that f and g are analytic in the region interior to and on γ and $f(z) \neq 0$ on γ. What is the value of the integral?

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} g(z) dz.$$