복소함수론 중간고사 , 1999년 4월 14일

1. Find all values of
a) $(-1)^{\frac{1}{6}}$
b) i^i

2. Let a function $f(z)$ be analytic in a domain D. Prove that if $|f(z)| = c$, where c is a constant, then f is a constant.

3. Tell whether or not $u(x, y)$ is harmonic. If harmonic find the harmonic conjugate of u.
a) $u(x, y) = \sinh x \sin y$
b) $u(r, \theta) = \ln r$

4. Let $w = f(z) = \sin z$. Find and sketch in the w plane the image of the line segment : $z = x + iy$ where $-\pi \leq x \leq \pi, \quad y = 2$.

5. Let C be the circle $z = 1 + 5e^{it}, \quad 0 \leq t \leq 2\pi$. Evaluate
a)$\int_C \frac{dz}{z-1}$
b) $\int_C \frac{dz}{(z-1)^n}$, where n is a positive integer, $n \geq 2$

6. Let C be a contour defined by $x = t^2, \quad y = \cos(\pi t) - 1$, $0 \leq t \leq 1$. Evaluate $\int_C (iz + 2)^4 dz$.

7. State the Cauchy-Goursat theorem and prove the theorem assuming that the function is C^1.

8. Let C_R be the circle $|z| = R, \quad R > 1$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\text{Log } z}{z^2} dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right)$$

and hence that the value of this integral approaches zero as R tends to infinity.

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9. Show that if z lies on the circle $|z| = 2$, then

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}.$$

10. Show that

$$1+ \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left((2n+1)\theta/2\right)}{2\sin(\theta/2)}, \quad (0 < \theta < 2\pi).$$

11. Find all roots of $e^{e^z} = 1$.

12. Let $a_1, \ldots, a_n$ be the distinct roots of $z^n = b, \quad n \geq 2$. Show that

$$a_1 + \cdots + a_n = 0.$$

13. Let $f(z)$ be analytic on a domain D and $z(t), \quad a \leq t \leq b$, be a contour in D. Let $w(t) := f(z(t))$.

a) Show that $w'(t) = f'(z(t))z'(t)$.

b) If f has an antiderivative in D then the integral $\int_C f(z)dz$ depends only on the end points of C.

14. Let $f = u + iv$ be analytic in a domain D and that $f' \neq 0$. Prove that a level curve of u is perpendicular to a level curve of v at every intersecting point.

15. Show that if u is harmonic in a domain D then $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in D.

16. Consider a 2-dimensional flow of fluid whose velocity vector V at (x, y) is $V = (u(x, y), v(x, y))$. Prove that if the fluid is incompressible and irrotational then the complex function $u - iv$ is analytic.

17. Suppose that $f(z)$ is a one-to-one analytic function of a domain D onto a domain $f(D)$ and that $f'(z) \neq 0, \quad \forall z \in D$. Show that the linear magnification at z is $|f'(z)|$ (i.e., an element of arc ds in the z-plane is multiplied by $|f'(z)|$ in the w-plane); also show that the area of $f(D)$ is

$$A = \iint_D |f'(z)|^2 dxdy.$$

18. Let C be a contour from $z = -1$ to $z = 1$ that lies in the upper half plane. We consider the principal branch of $z^{1/3}, \quad -\pi < Arg z < \pi$. Evaluate $\int_C z^{1/3} dz$