복소해석학 학기말 고사

2004. 12.17, 200점 만점

1 (20) State the following theorems precisely:
a) Morera's theorem
b) Mittag-Leffler theorem
c) Weierstrass theorem on infinite product
d) Riemann mapping theorem

2 (20) Let $u(r,\theta)$ be a harmonic function on $r<1$ with boundary values $\lim_{r\to 1} u(r,\theta)=\phi(\theta)$, where ϕ is a piecewise continuous function for $-\pi\le\theta\le\pi$.
a) Express $u(r,\theta)$ as an integral.
b) Find $u(r,\theta)$, when ϕ is given as

$$\phi(\theta)=\begin{cases}1, & \text{for } 0<\theta<\pi\\ -1, & \text{for } -\pi<\theta<0.\end{cases}$$

3 partial fractions (30)
a) Show that $\sum_{-\infty}^{\infty}\frac{1}{(z-n)^2}$ converges uniformly on each compact set after omitting finitely many terms.
b) Show that $\frac{\pi^2}{\sin^2\pi z}=\sum_{-\infty}^{\infty}\frac{1}{(z-n)^2}$.
c) Find the (infinite) partial fraction as in the Mittag-Leffler theorem for $\pi\cot(\pi z)$ and justify your answer.

4 (20) Find an entire function and tell its genus:
a) simple zeros at $\sqrt{n}$, $\quad n=1,2,\cdots$ and no other zeros.
b) simple zeros at $n(\log n)^2$, $\quad n=1,2,\cdots$ and no other zeros.

5 (10) If $f(z)$ is holomorphic in $|z|\le 1$ and satisfies $|f|=1$ on $|z|=1$, show that $f(z)$ is rational.

6 (10) Show that f is a holomorphic one-to-one mapping of a domain Ω onto a domain $\tilde{\Omega}$ then f is a conformal map, i.e., f' is nowhere zero.

7 (10) Show that $\int_{|\zeta-a|=1}\frac{d\zeta}{(\zeta-a)^k(\zeta-z)}=0$, for any z with $|z-a|<1$ and for any integer $k\ge 1$.

뒷면에 계속

8 (30)

a) Prove that the set $A^2(\Omega)$ of L^2 holomorphic functions on Ω is a closed subspace of $L^2(\Omega)$.

b) Find the Bergman kernel for the disk $|z| < R$ of radius R.

c) Evaluate

$$\frac{1}{\pi}\int_{|z|<1} \frac{\sin z dA(z)}{(1-i\bar{z})^2}.$$

9 (20) Let ϕ be a compactly supported complex-valued smooth function on $\mathbf{C}$.

a) For any locally integrable complex-valued function $K(w)$ on $\mathbf{C}$ and let

$$u(z) = \int_{\mathbf{C}} \phi(z+w)K(w)dA(w).$$

Show that

$$\frac{\partial u}{\partial \bar{z}}(z) = \int_{\mathbf{C}} \frac{\partial \phi}{\partial \bar{z}}(z+w)K(w)dA(w).$$

b) Now define a function u by

$$u(z) = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\phi(\zeta)}{\zeta - z} dA(\zeta).$$

Show that $\frac{\partial u}{\partial \bar{z}}(z) = \phi(z)$.

10 (30)

a) For $s \in \mathbf{C}$ and $z \in \mathbf{C} \setminus (-\infty, 0]$ define z^s by

$$z^s = e^{s \log z},$$

where log denotes the principal branch of logarithm. For $\rho > 0$ define C_ρ by $z = \rho e^{i\theta}$, $0 \le \theta \le \pi/2$. Show that

$$\lim_{\rho \to 0+} \int_{C_\rho} z^s e^{iz} \frac{dz}{z} = \lim_{\rho \to \infty} \int_{C_\rho} z^s e^{iz} \frac{dz}{z} = 0 \qquad (0 < \sigma < 1).$$

Here σ denotes the real part of s.

b) Use the result of a) to show that

$$\lim_{A \to \infty} \int_0^A x^s \sin x \frac{dx}{x} = \Gamma(s) \sin\frac{\pi s}{2} \qquad (0 < \sigma < 1)$$

and

$$\lim_{A \to \infty} \int_0^A x^s \cos x \frac{dx}{x} = \Gamma(s) \cos\frac{\pi s}{2} \qquad (0 < \sigma < 1).$$

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