복소해석학 중간고사

2004. 10.20, 앞 장의 문제만 제출할 것. 150점 만점

1 (20) Let U be the open unit disk $|z| < 1$. For $a \in U$ consider the function $\phi(z) = \frac{z-a}{1-\bar{a}z}$.
a) Show that ϕ maps U into U and ∂U into ∂U.
b) Show that $\phi : U \to U$ is one-to-one and onto.

2 (10) Show that any linear (linear fractional) transformation which transforms the real axis into itself can be written with real coefficients.

3 Evaluate the integrals:
(10) a) $\int_C \frac{2z^2+5}{(z^2+1)^2} dz$, where C is the circle $|z+i| = 1$.
(10) b) $\int_{|z|=1} \frac{\sin z}{z^3} dz$.
(10) c) $\int_\gamma (2\bar{z} + \frac{z^2+z+1}{z-3}) dz$, where γ is the circle of radius 1 centered at $1+i$.
(20) d) $\int_0^\infty \frac{x \sin x}{x^2+1} dx$.

4 (10) Let U be the open unit disk and $f \in H(\bar{U})$. Suppose that $f(z) > 0$ for all z with $|z| = 1$. Prove that f is constant on $\bar{U}$.

5 (15) If $P(z)$ is a polynomial and if f is holomorphic on all of $\mathbb{C}$, and if there exists a real constant C such that $|f(z)| \le C\ |P(z)|$ for every $z \in \mathbb{C}$, then $f = c\ P$ for some $c \in \mathbb{C}$. Is there an analogous statement with P replaced by an arbitrary holomorphic function on all of $\mathbb{C}$?

6 (25) Let $u(r, \theta) = \log r$ defined on $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$.
a) Show that u is harmonic on $\mathbb{C}^*$.
b) Find $*du$.
c) Find the period(s) of $*du$, that is, find $\int_\gamma *du$ for each closed curve γ of the homology basis of $\mathbb{C}^*$.
d) Find a conjugate harmonic function that is defined locally.
e) Does there exist a harmonic conjugate that is globally defined? Explain why or why not.

7 (10) Let f be a meromorphic function on a simply connected open set $\Omega \subset \mathbb{C}$. Prove that f has a primitive that is meromorphic on Ω if and only if at each pole of f the residue of f is zero. (참고 primitive = 원시함수)

8 (10) Prove that for each non-empty open subset $\Omega \subset \mathbb{C}$, and for all points $a, b \in \mathbb{C} \backslash \Omega$ in the same connected component of $\mathbb{C} \backslash \Omega$, there exists a complex square root of $(z-a)(z-b)$, in other words, a function $f \in H(\Omega)$ such that $[f(z)]^2 = (z-a)(z-b)$ for every $z \in \Omega$.

9 Evaluate

a) $\sin i$

b) i^i.

10 State and prove Morera's theorem.

11 Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

12 (10) Let U be the unit disk $|z| < 1$. Suppose that $f \in H(\bar{U})$ and that $|f(z)| < 1$ for z with $|z| = 1$. Show that f has a fixed point in U, that is, there is a point $z \in U$ such that $f(z) = z$. How many fixed points are there in U?

13 Suppose that $f \in H(\bar{U})$ and that f has no zero on $|z| = 1$. If $\phi \in H(\bar{U})$ what is the value of the integral?

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)\phi(z)}{f(z)} dz.$$

14 $f \in H(\bar{U})$ and f is bounded. Then $\forall \zeta \in U$

$$f(\zeta) = \frac{1}{\pi} \iint_U \frac{f(\zeta)}{(1 - \bar{z}\zeta)^2} dxdy.$$