# SMOOTH FUNCTIONS VANISHING ON A SUBMANIFOLD

**Lemma.** *Let $(t,x)$, where $t=(t_1,\cdots,t_d)$, $x=(x_1,\cdots,x_m)$, be the standard coordinates of $\mathbb{R}^{d+m}$. Suppose that $f$ is a $C^\infty$ function defined on a neighborhood of the origin such that $f(0,x)=0$. Then $f(t,x)=\sum_{j=1}^{d} t_j g^j(t,x)$, for some $C^\infty$ functions $g^1,\cdots,g^d$ defined on a smaller neighborhood of the origin.*

Proof

$$
\begin{aligned}
f(t,x) &= \int_0^1 \frac{\partial}{\partial \tau} f(\tau t, x)\, d\tau \\
&= \int_0^1 \sum_{j=0}^{d} t_j f_j(\tau t, x)\, d\tau, \qquad \text{where } f_j = \frac{\partial f}{\partial t_j} \\
&= \sum_{j=1}^{d} t_j \int_0^1 f_j(\tau t, x)\, d\tau.
\end{aligned}
$$

Let $g^j(t,x)=\int_0^1 f_j(\tau t,x)d\tau$, for each $j=1,\cdots,d$. Then it is standard to show that $g^j$ are $C^\infty$. $\square$