GENERALIZED FROBENIUS THEOREM WITH SINGULAR TORSIONS

On $\mathbb{R}^3 = \{(x, y, z)\}$ consider a 1-form

$$\theta = dz + f(x, y, z)dy, \tag{1}$$

where $d(x, y.z)$ is a smooth (C^∞) real valued function defined on an open neighborhood of the origin. We are concerned with the existence of integral manifolds of (1). Suppose that M is an integral manifold of (1). Since $\theta|_M = 0$ we have $(d\theta)|_M = 0$. Now

$$\begin{aligned} d\theta &= (f_x dx + f_y dy + f_z dz) \wedge dy \\ &= f_x dx \wedge dy, \qquad \mod \theta. \end{aligned}$$

The obstruction to the existence of integral manifolds is the torsion

$$T = f_x.$$

If T is identically zero then by the Frobenius theorem there exists a 1-parameter family of integral manifolds.

In order construct examples with singular torsion sets we set

$$T = f_x = z(z - g(x, y)) = z^2 - zg(x, y). \tag{2}$$

I want $z = 0$ is the only integral manifold, so that we require

$$\begin{cases} f(x, y.0) = 0 \\ f_x = z^2 - zg(x, y). \end{cases}$$

Second condition implies that

$$f(x, y.z) = z^2 x - zG(x, y),$$

where $G_x = g$. Now any pair (G, g) with $G_x = g$ yields the torsion (2).

Example 1. *$G(x, yz) = x^2$, $\quad g(x, y) = 2x$: Let*

$\theta = dz + (z^2x - zx^2)dy$. Then $d\theta \equiv (z^2 - 2zx)dx \wedge dy$, mod θ. Therefore, $T = z(z - 2x)$. The zero set of T is two planes intersecting along y-axis, among which the plane $z = 0$ is an integral manifold.

Example 2. *Let $f_x = z(z^2 - x^2 - y^2)$, so that $f(x, y, z) = z^3x - zx^3/3 - zy^2x$. Then the zero set of the torsion is given by $z(z^2 - x^2 - y^2) = 0$. This variety is the union of the plane $z = 0$ and the cone $z^2 - x^2 - y^2 = 0$. $z = 0$ is an integral manifold.*

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