Class number 1 criteria for real quadratic fields of Richaud-Degert type

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In this paper we develop two ways of computing special values of zeta function attached to a real quadratic field. Comparing these values we obtain various class number 1 criteria for real quadratic fields of Richaud-Degert type.

Let k be a real quadratic field. It is an interesting problem, especially in class number one problem, to find a necessary and sufficient condition that k has class number one. H. Yokoi [11] proved the following result.

> Let $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + 1$, be a real quadratic field. Then the class number of k is 1 if and only if $n^2 - t(t+1)$, $1 \leq t \leq n-1$, is a prime.

His proof is an algebraic proof. In this paper we shall give an analytic proof of the above theorem. Actually we shall develop two ways of computing special values of zeta function attached to a real quadratic field. Comparing these values, we shall obtain various class number 1 criteria for real quadratic fields of Richaud-Degert (R-D) type.

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1 Special values of zeta functions

Let k be a totally real quadratic field and $\zeta_k(s)$ be its Dedekind zeta function. Using the finite dimensionality of the space of elliptic modular forms of weight h, C.L. Siegel [10] developed a method of computing $\zeta_k(1-2n)$, where n is a positive integer. By specializing Siegel's formula for a real quadratic field, we obtain the following result.

Theorem 1.1 *Let k be a real quadratic field with discriminant D. Then*

$$\zeta_k(-1) = \frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D (\mathrm{mod}\, 4)}} \sigma_1\left(\frac{D-b^2}{4}\right),$$

where $\sigma_1(r)$ denote the sum of divisors of r.

Proof: See [13]. □

However there is another method of computing special values of $\zeta_k(s)$ if k is a real quadratic field due to H. Lang.

Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant D and A an ideal class of k. Let ϵ be the fundamental unit of k and $\boldsymbol{a}$ be any integral ideal belonging to A^{-1}. Let r_1, r_2 be an integral basis of $\boldsymbol{a}$ and r_1', r_2' be their conjugates. We put

$$\delta(\boldsymbol{a}) = r_1 r_2' - r_1' r_2. \tag{1}$$

Since $\epsilon r_1, \epsilon r_2$ are also an integral basis of $\boldsymbol{a}$, we can find an integral matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying

$$\epsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{2}$$

Now we can state Lang's formula.

Theorem 1.2 *By keeping the above notation, we have*

$$\begin{aligned} \zeta_k(-1, A) \;\; = \;\; & \frac{sgn\, \delta(\boldsymbol{a}) r_2 r_2'}{360 N(\boldsymbol{a}) c^3} \{(a+d)^3 - 6(a+d)N(\epsilon) \\ & -240c^3\,(sgn\, c) S^3(a,c) + 180ac^3\,(sgn\, c) S^2(a,c) \\ & -240c^3\,(sgn\, c) S^3(d,c) + 180dc^3\,(sgn\, c) S^2(d,c)\}, \end{aligned}$$

where $S^i(a,c) = S_4^i(a,c)$ denote the generalized Dedekind sum.

Proof: This is a main theorem of [6]. □

To use Lang's formula, we need to compute a, b, c, d and generalized Dedekind sums.

Lemma 1.3 *Put* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then*

$$a = tr\left(\frac{r_1 r_2' \epsilon}{\delta(\boldsymbol{a})}\right), \qquad b = tr\left(\frac{r_1 r_1' \epsilon'}{\delta(\boldsymbol{a})}\right)$$
$$c = tr\left(\frac{r_2 r_2' \epsilon}{\delta(\boldsymbol{a})}\right), \qquad d = tr\left(\frac{r_1 r_2' \epsilon'}{\delta(\boldsymbol{a})}\right).$$

Furthermore, $\det M = N(\epsilon)$ *and* $bc \neq 0$.

Proof: By (2) and its conjugate, we have

$$\begin{bmatrix} \epsilon r_1 \\ \epsilon r_2 \end{bmatrix} = M \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad \begin{bmatrix} \epsilon' r_1' \\ \epsilon' r_2' \end{bmatrix} = M \begin{bmatrix} r_1' \\ r_2' \end{bmatrix}. \tag{3}$$

Write (3) in matrix form to get

$$\begin{bmatrix} \epsilon r_1 & \epsilon' r_1' \\ \epsilon r_2 & \epsilon' r_2' \end{bmatrix} = M \begin{bmatrix} r_1 & r_1' \\ r_2 & r_2' \end{bmatrix}. \tag{4}$$

We get the desired result by multiplying

$$\begin{bmatrix} r_1 & r_1' \\ r_2 & r_2' \end{bmatrix}^{-1} = \frac{1}{\delta(\boldsymbol{a})} \begin{bmatrix} r_2' & -r_1' \\ -r_2 & r_1 \end{bmatrix}$$

on both sides of (4). □

Applying reciprocity law for generalized Dedekind sums (see, for example, [1, 2]), we have the following results.

Lemma 1.4 *Let m be a positive integer. Then we have*

(i) $S^3(\pm 1, m) = \pm \dfrac{-m^4 + 5m^2 - 4}{120 m^3}$,

(ii) $S^2(\pm 1, m) = \dfrac{m^4 + 10m^2 - 6}{180m^3}$.

Proof: See [4]. □

Lemma 1.5 *Let m be a positive even integer. Then we have*

(i) $S^3(m+1, 2m) = S^1(m+1, 2m) = \dfrac{-m^4 + 50m^2 - 4}{120(2m)^3}$,

(ii) $S^3(m-1, 2m) = -S^1(m+1, 2m) = \dfrac{m^4 - 50m^2 + 4}{120(2m)^3}$,

(iii) $S^2(m-1, 2m) = -S^2(m+1, 2m) = \dfrac{m^4 + 100m^2 - 6}{180(2m)^3}$.

Proof: See [4]. □

2 Main theorem

In this section, we compare the values $\zeta_k(-1)$ and $\zeta_k(-1, A)$ and derive our main theorem. We start from a definition.

Definition 2.1 Let $d = n^2 + r$, $d \neq 5$, be a positive square free integer satisfying the conditions

$$r | 4n \quad \text{and} \quad -n < r \leq n.$$

In this situation, the real quadratic field $k = \mathbb{Q}(\sqrt{d})$ is called a real quadratic field of Richaud-Degert (R-D) type.

Proposition 2.2 *Let $k = \mathbb{Q}(\sqrt{d})$, $d > 0$, be a real quadratic field of R-D type. Then the fundamental unit ϵ and its norm $N(\epsilon)$ are given as follows :*

$$\epsilon = n + \sqrt{n^2 + r}, \quad N(\epsilon) = -sgn\, r \quad if |r| = 1,$$
$$\epsilon = \frac{n + \sqrt{n^2 + r}}{2}, \quad N(\epsilon) = -sgn\, r \quad if |r| = 4,$$

and

$$\epsilon = \frac{2n^2 + r}{|r|} + \frac{2n}{|r|}\sqrt{n^2 + r}, \quad N(\epsilon) = 1 \quad if |r| \neq 1, 4.$$

Proof: See Degert [3]. □

Theorem 2.3 *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of R-D type and let D denote the discriminant of k. Then, for each case, the following equality for $\zeta_k(-1)$ is equivalent to the condition $h_k = 1$.*

I. $d = n^2 + r \equiv 2, 3$ *(mod 4)*

(i) $|r| \neq 1, 4$

$$\zeta_k(-1) = \frac{4n^3(r^2+1) + 2nr(3r^2+5r+3)}{180r^2}$$

(ii) $|r| = 1$

$$\zeta_k(-1) = \frac{4n^3 + 5n \pm 6n}{180}$$

II. $d = n^2 + r \equiv 1$ *(mod 4)*

(i) $|r| \neq 1, 4$

$$\zeta_k(-1) = \frac{2n^3(r^2+1) + n(3r^3+50r^2+3r)}{720r^2} \quad \text{if } n \text{ even}$$

$$\zeta_k(-1) = \frac{2n^3(r^2+16) + n(3r^3+20r^2+48r)}{720r^2} \quad \text{if } n \text{ odd}$$

(ii) $|r| = 4$ *(hence n odd)*

$$\zeta_k(-1) = \frac{n^3 + 5n \pm 6n}{360}$$

(iii) $|r| = 1$ *(hence $r = 1$ and n even)*

$$\zeta_k(-1) = \frac{n^3 + 14n}{360}.$$

Proof: Let C denote the ideal class of principal ideals. Then

$$\zeta_k(-1) \geq \zeta_k(-1, C)$$

and equality holds if and only if $h_k = 1$. We prove the theorem by computing $\zeta_k(-1, C)$. We give detailed computation only for the case I(i), since the other cases are similar to this case.

Now assume that $d = n^2 + r \equiv 2, 3 \pmod 4$ and $|r| \neq 1, 4$. In this case, $D = 4d$ and $r_1 = \sqrt{n^2 + r}$, $r_2 = 1$ form an integral basis for $\mathcal{O}_k$. Hence we can take $\boldsymbol{a} = \mathcal{O}_k = [r_1, r_2]$ in Theorem 1.2. By Proposition 2.2,

$$\epsilon = \frac{2n^2 + r}{|r|} + \frac{2n}{|r|}\sqrt{n^2 + r}$$

is the fundamental unit of k and $N(\epsilon) = 1$. By Lemma 1.3, we have

$$\begin{aligned} \epsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} &= (\frac{2n^2 + r}{|r|} + \frac{2n}{|r|}\sqrt{n^2 + r}) \begin{bmatrix} \sqrt{n^2 + r} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2n^2+r}{|r|} & \frac{2n(n^2+r)}{|r|} \\ \frac{2n}{|r|} & \frac{2n^2+r}{|r|} \end{bmatrix} \begin{bmatrix} \sqrt{n^2 + r} \\ 1 \end{bmatrix}. \end{aligned}$$

Note that

$$\frac{2n^2 + r}{|r|} = n\frac{2n}{|r|} + \operatorname{sgn} r \equiv \operatorname{sgn} r \pmod{\frac{2n}{|r|}}.$$

Now put $\eta = \operatorname{sgn} r$. Then, by Lemma 1.4,

$$\begin{aligned} 240c^3 \operatorname{sgn} c S^3(a, c) &= 240c^3 S^3(\eta, c) = -\frac{8n}{r^4}(4n^4 - 5n^2r^2 + r^4), \\ 180c^3 \operatorname{sgn} c S^2(a, c) &= 180ac^3 S^2(\eta, c) = -\frac{2n}{r^5}(2n^2 + r)(8n^4 + 20n^2r^2 - 3r^4), \end{aligned}$$

and

$$(a + d)^3 - 6(a + d)N(\epsilon) = 8\eta\frac{2n^2 + r)^3}{r^3} - 12\eta\frac{2n^2 + r}{r}.$$

By substitution these results to Theorem 1.2, we get

$$\zeta_k(-1, C) = \frac{4n^3(r^2 + 1) + 2nr(3r^2 + 5r + 3)}{180r^2}.$$

□

Comparing Theorem 1.1 and Theorem 2.3 we obtain

Theorem 2.4 *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of R-D type and D be the discriminant of k. Then, for each case, the following equality is equivalent to the condition that $h_k = 1$.*

I. $d = n^2 + r \equiv 2, 3$ *(mod 4)*

(i) $|r| \neq 1, 4$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1\left(\frac{D-b^2}{4}\right) = \frac{4n^3(r^2+1) + 2nr(3r^2+5r+3)}{180r^2}$$

(ii) $|r| = 1$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1\left(\frac{D-b^2}{4}\right) = \frac{4n^3 + 5n \pm 6n}{180}$$

II. $d = n^2 + r \equiv 1$ *(mod 4)*

(i) $|r| \neq 1, 4$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1\left(\frac{D-b^2}{4}\right) = \frac{2n^3(r^2+1) + n(3r^3+50r^2+3r)}{720r^2}$$

if n even

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1\left(\frac{D-b^2}{4}\right) = \frac{2n^3(r^2+16) + n(3r^3+20r^2+48r)}{720r^2}$$

if n odd

(ii) $|r| = 4$ *(hence n odd)*

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1\left(\frac{D-b^2}{4}\right) = \frac{n^3 + 5n \pm 6n}{360}$$

(iii) $|r| = 1$ *(hence $r = 1$ and n even)*

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1\left(\frac{D-b^2}{4}\right) = \frac{n^3 + 14n}{360}$$

3 Class number 1 criteria for real quadratic fields of Richaud-Degert type

In this section we shall apply Theorem 2.4 to obtain class number 1 criteria for all real quadratic fields of R-D type. Recall that $k = \mathbb{Q}(\sqrt{d})$ is a real quadratic field of R-D type if $d(\neq 5)$ is a square free integer of the form $n^2 + r$ such that $r|4n$, $-n < r \leq n$. We devide the situation into two cases.

Case I. $d = n^2 + r \equiv 2, 3 \pmod 4$

Corollary 3.1 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 - 1$, $n > 1$. *Then* $h_k > 1$.

Corollary 3.2 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 + 1$, $n > 1$. *Then* $h_k > 1$.

Corollary 3.3 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 - 2$, $n \geq 1$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad \begin{array}{l} 4n^2 + 4n - 1 - 4t^2 \, (0 \leq t \leq n), \\ 2n^2 + 2n - 1 - 2t^2 + 2t \, (1 \leq t \leq n) \quad \textit{are primes.} \end{array}$$

Corollary 3.4 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 - 2$, $n \geq 2$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad \begin{array}{l} 2n^2 - 1 - 2t^2 \, (0 \leq t \leq n-1), \\ 4n^2 - 3 - 4t^2 + 4t \, (1 \leq t \leq n) \quad \textit{are primes.} \end{array}$$

Corollary 3.5 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 - 2$, $n \geq 1$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad \begin{array}{l} 2n^2 + 2n + 1 - 2t(t+1) \, (0 \leq t \leq n-1), \\ (2n+1)^2 + 2 - 4t^2 \, (0 \leq t \leq n) \quad \textit{are primes.} \end{array}$$

Corollary 3.6 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + 2$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad \begin{array}{l} 2n^2 + 1 - 2t^2 \, (0 \leq t \leq n-1), \\ 4n^2 + 2 - (2t-1)^2 \, (1 \leq t \leq n) \quad \textit{are primes.} \end{array}$$

Corollary 3.7 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 + r$, $r \equiv 1(4)$, $r|2n+1$, $r > 1$. *Write* $2n + 1 = rm$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad \begin{array}{l} r, r^2m^2 + r - t^2 \, (1 \leq t \leq m, r \nmid t), \\ rm^2 + 1 - rs^2 \, (0 \leq s \leq m-1) \quad \textit{are primes.} \end{array}$$

Corollary 3.8 *Let* $k = \mathbb{Q}(\sqrt{d}),\ d = (2n+1)^2 + 2r,\ r \equiv 1, 3(4),\ r|2n+1,$ $r \neq 1$. *Then* $h_k > 1$.

Corollary 3.9 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 - r$, $r \equiv 3(4)$, $r|2n+1$, $r > 1$. *Write* $2n+1 = rm$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad r, r^2m^2 - r - t^2\,(1 \leq t \leq rm-1, r \not| t), \\ rm^2 - 1 - rs^2\,(0 \leq s \leq m-1) \quad \textit{are primes.}$$

Corollary 3.10 *Let* $k = \mathbb{Q}(\sqrt{d}),\ d = (2n+1)^2 - 2r,\ r \equiv 1, 3(4),\ r|2n+1,$ $r \neq 1$. *Then* $h_k > 1$.

Corollary 3.11 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + r$, $r \equiv 3(4)$, $r|n$. *Write* $n = rm$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad r, 4r^2m^2 + r - t^2\,(1 \leq t \leq 2rm, r \not| t), \\ 4rm^2 + 1 - rs^2\,(0 \leq s \leq 2m-1) \quad \textit{are primes.}$$

Corollary 3.12 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + 2r$, $r \equiv 1, 3(4)$, $r|n$, $r \neq 1$. *Then* $h_k > 1$.

Corollary 3.13 *Let* $k = \mathbb{Q}(\sqrt{d}),\ d = (2n)^2 - r,\ r \equiv 1, 3(4),\ r|n,\ r > 1$. *Write* $n = rm$. *Then*

$$h_k = 1 \quad \Leftrightarrow \quad r, 4r^2m^2 - r - t^2\,(1 \leq t \leq 2rm-1, r \not| t), \\ 4rm^2 - 1 - rs^2\,(0 \leq s \leq 2m-1) \quad \textit{are primes.}$$

Corollary 3.14 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 - 2r$, $r \equiv 1, 3(4)$, $r|n$, $r > 1$. *Then* $h_k > 1$.

<u>Case II. $d = n^2 + r \equiv 1 \pmod 4$</u>

Corollary 3.15 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + 1$. *Then*

$$h_k = 1 \Leftrightarrow n^2 - t(t+1),\ 1 \leq t \leq n-1,\ \textit{is prime.}$$

Corollary 3.16 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 + 4$. *Then*

$$h_k = 1 \Leftrightarrow n^2 + n + 1 - t(t+1),\ 0 \leq t \leq n-1,\ \textit{is prime.}$$

Corollary 3.17 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 - 4$, $n > 1$. Then*

$$h_k = 1 \Leftrightarrow n^2 + n - 1 - t(t+1),\ 0 \leq t \leq n-1, \text{ is a prime.}$$

Corollary 3.18 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 - r$, $r \equiv 3$, (mod 4), $r|n$, $r > 1$. Write $n = rm$. Then*

$$\begin{aligned} h_k = 1 \quad \Leftrightarrow \quad & r, n + \frac{r+1}{4}, n - \frac{r+1}{4}, \\ & n^2 - \frac{(2t+1)^2 + r}{4}\,(0 \leq t \leq n-1, r \not| 2t+1), \\ & rm^2 + \frac{r+1}{4} - r(k^2+k)\,(0 \leq k \leq m-1) \quad \text{are primes.} \end{aligned}$$

Corollary 3.19 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + r$, $r \equiv 1$ (mod 4), $r|n$, $r > 1$. Write $n = rm$. Then*

$$\begin{aligned} h_k = 1 \quad \Leftrightarrow \quad & r, n - \frac{1-r}{4}, n + \frac{1-r}{4}, \\ & n^2 - \frac{1-r}{4} - t^2 - t\,(0 \leq t \leq n-1, r \not| 2t+1), \\ & rm^2 - \frac{r-1}{4} - r(k^2+k)\,(0 \leq k \leq m-1) \quad \text{are primes.} \end{aligned}$$

Corollary 3.20 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 + 4r$, $r|2n+1$, $r > 1$. Write $2n+1 = rm$. Then*

$$\begin{aligned} h_k = 1 \quad \Leftrightarrow \quad & n^2 + n + r - t^2 - t\,(0 \leq t \leq n, r|n-t) \quad \text{and} \\ & r(\frac{m^2}{4} - k^2 - k - 1) + 1\,(0 \leq k \leq \frac{m-3}{2}) \quad \text{are primes.} \end{aligned}$$

Corollary 3.21 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 - 4r$, $r|2n+1$, $r > 1$. Write $2n+1 = rm$. Then*

$$\begin{aligned} h_k = 1 \quad \Leftrightarrow \quad & n^2 + n - r - t^2 - t\,(0 \leq t \leq n, r|n-t) \quad \text{and} \\ & r(\frac{m^2}{4} - k^2 - k - 1) - 1\,(0 \leq k \leq \frac{m-3}{2}) \quad \text{are primes.} \end{aligned}$$

We shall give the proof of Corollary 3.7. The other cases are similar to this.

Proof of Corollary 3.7: We have $D = 4d$. By Siegel's computation,

$$\begin{aligned}
\zeta_k(-1) &= \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D (\mathrm{mod}\, 4)}} \sigma_1\left(\frac{D-b^2}{4}\right) \\
&= \frac{1}{60}\left\{2\sum_{t=1}^{rm} \sigma_1(r^2m^2 + r - t^2 + \sigma_1(r^2m^2 + r)\right\} \\
&= \frac{1}{60}\left\{2\sum_{\substack{t=1,\cdots,rm \\ r \nmid t}} \sigma_1(r^2m^2 + r - t^2)\right. \\
&\qquad \left. +2\sum_{\substack{t=1,\cdots,rm \\ r|t}} \sigma_1(r^2m^2 + r - t^2) + \sigma_1(r^2m^2 + r)\right\}
\end{aligned}$$

(For the case $r|t$, write $t = rs$, $1 \le s \le m-1$.
Then $r^2m^2 + r - t^2 = r(rm^2 + 1 - rs^2)$)

$$\begin{aligned}
&\ge \frac{1}{60}\left\{2\sum_{t=1}^{rm}(1 + r^2m^2 + r - t^2) + 1 + r^2m^2 + r\right\} \\
&\quad + \frac{1}{60}\left\{2\sum_{s=1}^{m-1}(r + rm^2 + 1 - rs^2) + r + rm^2 + 1\right\} \\
&= \frac{4r^3m^3 + 6r^2m + 5rm + 3r + 3}{180} + \frac{4rm^3 + 5rm + 6m - 3r - 3}{180} \\
&= \frac{4r^3m^3 + 4rm^3 + 6r^2m + 10rm + 6m}{180} \\
&= \zeta_k(-1, C)
\end{aligned}$$

Note that equality holds if and only if r, $r^2m^2 + r - t^2$ $(1 \le t \le rm,\ r \nmid t)$, $rm^2 + 1 - rs^2$ $(0 \le s \le m-1)$ are primes. □

Remark 1. There is an interesting result concerning the number of d's satisfying our condition, i.e. the number of real quadratic fields of Richaud-

Degert type of class number one. It is known (see, for example, [9] [12]) that there are exactly 39 real quadratic fields, i.e.

$$Q(\sqrt{d}); d = 2, 3, 6, 7, 11, \ldots, 1253, 1293, 1757$$

of Richaud-Degert type of class number one with one more possible exception, and under the assumption of the generalized Riemann Hypothesis this is true without any exception. The problem of finding all such d's without assuming the generalized Riemann Hypothesis is still open.

Remark 2. H. Yokoi [11] obtained Corollary 3.15 by using algebraic method. H. Lu [7, 8] obtained the above corollaries by using the theory of continued fractions. M. Kobayashi [5] obtained stronger condition that a real quadratic field to be of class number 1.

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