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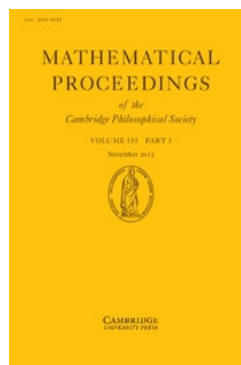
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***p*-adic *L*-functions over the false Tate curve extensions**

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Abstract

Let f be a primitive modular form of CM type of weight k and level $\Gamma_0(N)$. Let p be an odd prime which does not divide N , and for which f is ordinary. Our aim is to p -adically interpolate suitably normalized versions of the critical values $L(f, \rho\chi, n)$, where $n = 1, 2, \dots, k - 1$, ρ is a fixed self-dual Artin representation of M_∞ defined by (1.1) below, and χ runs over the irreducible Artin representations of the Galois group of the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} . As an application, if $k \geq 4$, we will show that there are only finitely many χ such that $L(f, \rho\chi, k/2) = 0$, generalizing a result of David Rohrlich. Also, we conditionally establish a congruence predicted by non-commutative Iwasawa theory and give numerical evidence for it.



1. Introduction

The main conjectures of Iwasawa theory provide a general method for relating the critical values of complex L -functions to the arithmetic of a motive J , over certain infinite p -adic Lie extension M_∞ over a fixed base field K , which is always assumed to be a finite extension of \mathbb{Q} . A key part of this main conjecture is to prove the existence of the relevant p -adic L -function for J over M_∞ . Unlike anything that is known for the theory of complex L -functions, this p -adic L -function is conjectured to lie in the K_1 group of a certain canonical localization of the Iwasawa algebra of the Galois group of M_∞ over K , and, as was first pointed out by Kato [11], this leads one to study certain mysterious congruences between cyclotomic L -functions attached to the twist of J by arbitrary Artin representations of the Galois group of M_∞ over K . So far, these non-abelian p -adic L -functions have only been proven to exist when J is the Tate motive, M_∞ is totally real and contains the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} , and the finally the relevant Iwasawa $\mu = 0$ conjecture is valid. However, several authors [1, 2, 5, 6, 7] have obtained partial results about the existence of these p -adic L -functions, when J is the motive attached to a primitive modular form and

$$M_\infty = \bigcup_{n \geq 1} \mathbb{Q}(\mu_{p^n}, m^{1/p^n}) \tag{1.1}$$

for an odd prime p and a p -th-power-free integer m . This type of extension is called a false Tate curve extension. In this paper, we will consider a family of modular forms of

general weight $k \geq 2$ obtained from CM elliptic curves. We will establish an analogue of David Rohlich’s result on the non-vanishing of central L -values when the k is at least 4, and conditionally prove in Theorem 9.1 the first case of Kato’s congruences for certain false Tate curve extensions.

2. Statement of main results

Let $\overline{\mathbb{Q}}$ be an algebraic closure of the field of rational numbers \mathbb{Q} . For a number field $K \subset \overline{\mathbb{Q}}$, let \mathcal{O}_K be the ring of integers of K and \mathbb{A}_K be the ring of adèles of K . For a place v of K and an element $x \in K$, we denote by x_v the image of x under the natural inclusion $K \rightarrow K_v$. We fix, once and for all, embeddings $i_v: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_v$ for all places v of \mathbb{Q} . In particular, denoting by c the complex conjugation, c acts on $\overline{\mathbb{Q}}$, which is in fact independent of the fixed embedding i_∞ when restricted to totally imaginary quadratic extensions of totally real fields. Let \mathcal{K} be an imaginary quadratic field of class number one. Given an elliptic curve E/\mathbb{Q} with complex multiplication by $\mathcal{O}_\mathcal{K}$, one can associate the (adelic) grossencharacter

$$\psi_E: \mathbb{A}_\mathcal{K}^\times / \mathcal{K}^\times \longrightarrow \mathbb{C}^\times$$

whose infinity type is normalized as

$$\psi_E(x_\infty) = x^{-1}.$$

To lighten the notation, we will omit E from ψ_E and write $\psi = \psi_E$. We also let

$$w_\mathcal{K} := |\mathcal{O}_\mathcal{K}^\times / \{\pm 1\}|, \tag{2.1}$$

and make the following hypothesis.

HYPOTHESIS 1. k is a positive even integer such that $k - 1$ is prime to $w_\mathcal{K}$.

Since ψ^r has conductor when 1 whenever r is divisible by $2w_\mathcal{K}$, it follows from Hypothesis 1 that ψ^{k-1} has the same conductor as ψ . If we write τ_f for the ideal theoretic grossencharacter associated to an adelic grossencharacter τ , then we have $(\psi_f)^{k-1} = (\psi^{k-1})_f$ and we may write ψ_f^{k-1} without ambiguity. Put

$$f(z) = \sum_{\mathfrak{a}} \psi_f^{k-1}(\mathfrak{a}) e^{2\pi i N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{a})z} = \sum_n a_f(n) e^{2\pi i n z},$$

where \mathfrak{a} runs over the integral ideals of \mathcal{K} prime to the conductor of ψ , and $N_{\mathcal{K}/\mathbb{Q}}$ is the norm map from \mathcal{K} to \mathbb{Q} . Since E is defined over \mathbb{Q} , we have $\psi(x^c) = \psi(x)^c$. In particular, we have $a_f(n) \in \mathbb{Z}$ for all n . It is well known that f is a Hecke eigenform of conductor $N = N_E$, the conductor of E ; for example, see [12, theorem 4.8.2]. Now, one can attach to f and a rational prime ℓ , a two dimensional vector space $V_\ell(f)$ over $\overline{\mathbb{Q}}_\ell$ with continuous action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is unramified outside of $N\ell$. The action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_\ell(f)$ is characterized by the property that the trace of (geometric) Frobenius Frob_q is equal to $a_f(q)$ for every q relatively prime to $N\ell$. By an Artin representation θ , we mean a finite dimensional vector space $V(\theta)$ over $\overline{\mathbb{Q}}$ on which $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via a finite quotient. For an Artin representation θ , we simply write $V_\ell(\theta) := V(\theta) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_\ell$ for each prime ℓ . For such θ and a prime q , we choose a prime ℓ relatively prime to qN and put

$$P_q(f, \theta, X) := \det \left(1 - \text{Frob}_q X; (V_\ell(f) \otimes_{\overline{\mathbb{Q}}_\ell} V_\ell(\theta))^{I_q} \right),$$

where I_q denotes the absolute inertia subgroup of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Also, we put

$$P_q(\theta, X) = \det(1 - \text{Frob}_q X; V(\theta)^{I_q}).$$

The coefficients of $P_q(\theta, X)$ and $P_q(f, \theta, X)$ are algebraic integers which are independent of the choice of ℓ . Then the *L*-function of f twisted by θ is defined as

$$L(f, \theta, s) = \prod P_q(f, \theta, q^{-s}),$$

as q runs over all finite primes of \mathbb{Q} . The above Euler product converges when $\text{Re}(s) > (k + 1)/2$ and conjectured to have analytic continuation to the entire complex plane. We denote by $\mathbf{1}$ the trivial Artin representation, and simply write

$$P_q(f, X) = P_q(f, \mathbf{1}, X).$$

Of course, we have

$$L(f, \mathbf{1}, s) = L(f, s) = \sum_n a_f(n)n^{-s}$$

provided that $\text{Re}(s) > (k + 1)/2$. Throughout, we fix a prime p that satisfies the following hypothesis.

HYPOTHESIS 2. *We have $(p, 2N) = 1$ and p splits in \mathcal{K} .*

It is well known that Hypothesis 2 implies that the *p*-adic Galois representation attached to E is ordinary at p , whence it follows easily that the Galois representation $V_p(f)$ is also ordinary at p . For an Artin representation θ and an integer n , write $d(\theta)$ for the dimension of θ , and $d^+(\theta, n)$ (resp. $d^-(\theta, n)$) for the dimension of the subspace of $V(\theta)$ on which the complex conjugation acts by multiplication by $(-1)^n$ (resp. by $(-1)^{n+1}$). Let ω_E be the Neron differential on the global minimal Weierstrass model of E over \mathbb{Q} . Let Ω_E^+ be the minimal positive period of ω_E , and Ω_E^- be the absolute value of least purely imaginary period of ω_E . Let $\epsilon_p(\theta) = \epsilon_p(\theta, \Psi, dx)$ be the epsilon factor of θ at p , normalized as in Section 6. We define the canonical period $\Omega_\infty^{\text{can}}(f, \theta, n)$ by

$$\Omega_\infty^{\text{can}}(f, \theta, n) = \epsilon_p(\theta)^{-1} (2\pi i)^{d(\theta)n} (\Omega_E^+)^{(k-1)d^+(\theta, n)} (\Omega_E^-)^{(k-1)d^-(\theta, n)}. \tag{2.2}$$

It is conjectured that the values

$$L_p(f, \theta, n) := \frac{L(f, \theta, n)}{\Omega_\infty^{\text{can}}(f, \theta, n)}, \tag{2.3}$$

where $n = 1, 2, \dots, k - 1$, are all algebraic numbers. We shall consider a special case when θ varies over an infinite family of Artin representations for which this conjecture is known, and our aim will be to study their *p*-adic interpolation. Let $m > 1$ be a *p*th power free integer, and put

$$F := \mathbb{Q}(\mu_p), \text{ and } M := F(m^{1/p}).$$

Then F is a Galois extension of \mathbb{Q} , and we denote the corresponding Galois groups by

$$\Delta = \text{Gal}(M/\mathbb{Q}), \quad \Delta_0 = \text{Gal}(M/F), \quad \text{and } \Delta_1 = \text{Gal}(F/\mathbb{Q}). \tag{2.4}$$

We fix a nontrivial character

$$\eta: \Delta_0 \longrightarrow \mathbb{C}^\times$$

and let

$$\rho := \text{Ind}_{\Delta_0}^{\Delta} \eta \quad \text{and} \quad \sigma := \text{Ind}_{\Delta_0}^{\Delta} \mathbf{1} \tag{2.5}$$

be the representation of Δ induced from η and $\mathbf{1}$ respectively. ρ is an irreducible representation of dimension $p - 1$, which is independent of the choice of η , while σ is the sum of the $p - 1$ characters of Δ_1 . In other words, σ is sum of Dirichlet characters whose conductor divides p . It is conjectured that the normalization (2.3) satisfies

$$L_p(f, \theta, n)^g = L_p(f, \theta^g, n) \tag{2.6}$$

for any element g in the absolute Galois group of \mathbb{Q} . When θ is of the form $\sigma \chi$ for an Artin representation χ of Γ , (2.6) follows from the classical theory of Eisenstein and Damerell, while the case $\theta = \rho \chi$ follows from a theorem of Shimura for the field $\mathbb{Q}(\mu_p)$. By Hypothesis 2, we can write

$$P_p(f, X) = 1 - a_f(p)X + p^{k-1}X^2 = (1 - \alpha X)(1 - \beta X) \tag{2.7}$$

where, by Hypothesis 2, one of the roots, say α , is a p -adic unit in \mathbb{Z}_p . For an ideal r of \mathbb{Z} , we denote by $\text{ord}_p(r)$ the exponent of $p\mathbb{Z}$ appearing in the decomposition of r into product of prime ideals. For an Artin representation θ of dimension $d(\theta)$, let $\mathfrak{f}(\theta)$ be the conductor of θ , $e_p(\theta) := \text{ord}_p(\mathfrak{f}(\theta))$, and θ^\wedge be the contragredient of θ . We write $\Gamma(s)$ for the classical gamma function. Following [3], for $n = 1, 2, \dots, k - 1$, we define

$$\mathcal{L}_p(f, \theta, n) := L_p(f, \theta, n) \Gamma(n)^{d(\theta)} \prod_{q|pm} P_q(f, \theta, q^{-n}) \times \frac{P_p\left(\theta^\wedge, \frac{p^{n-1}}{\alpha}\right)}{P_p\left(\theta, \frac{\alpha}{p^n}\right)} \times \left(\frac{p^{n-1}}{\alpha}\right)^{e_p(\theta)}. \tag{2.8}$$

Let \mathbb{Q}^{cyc} be the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} and put $\Gamma := \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$. We denote by μ_{p^n} the group of p^n -th roots of unity in $\overline{\mathbb{Q}}$, and write μ_{p^∞} for the union of all μ_{p^n} with $n \in \mathbb{N}$. The restriction map from $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ to $\text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$ maps $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$ isomorphically to Γ , and we will view Γ as a subgroup of $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$, and $\kappa : \Gamma \rightarrow \mathbb{Z}_p^\times$ be the restriction to Γ of the cyclotomic character giving the action of the absolute Galois group of \mathbb{Q} on μ_{p^∞} . Let \mathcal{I} be the p -adic completion of the valuation ring in the maximal unramified extension of \mathbb{Q}_p . We define the Iwasawa algebra $\mathcal{I}[[\Gamma]]$ of Γ with coefficients in \mathcal{I} by

$$\mathcal{I}[[\Gamma]] = \varprojlim \mathcal{I}[\Gamma/U],$$

where the inverse limit is taken over all open subgroups U of Γ . Let γ be the unique topological generator of Γ which maps to $1 + p$ by κ , and identify $\mathcal{I}[[T]] \cong \mathcal{I}[[\Gamma]]$ by sending γ to $1 + T$. We will prove that the values $\mathcal{L}_p(f, \rho \chi, n)$ admit a p -adic interpolation by an element of Iwasawa algebra as we vary χ among the finite order characters of Γ and $n = 1, 2, \dots, k - 1$.

THEOREM (Theorem 7.2). *There exists a p -adic period $\Omega_p(f, \rho) \in \mathcal{I}^\times$ and an element $H(\rho, T) \in \frac{1}{p}\mathcal{I}(\mu_p)[[T]]$ such that for every Dirichlet character χ of p -power order and conductor, we have*

$$\frac{H(\rho, \chi(1+p)(1+p)^n - 1)}{\Omega_p(f, \rho)} = \mathcal{L}_p(f, \rho \chi, n). \tag{2.9}$$

For the definition of the *p*-adic period $\Omega_p(f, \rho)$, we refer the reader to (7.1) and Section 3. As an application, we obtain a generalization of a result in [14] on nonvanishing of central values of twisted *L*-functions:-

THEOREM (Theorem 8.1). *Assume that $k \geq 4$. There are only finitely many Dirichlet characters χ of *p*-power conductor and order such that*

$$L(f, \rho\chi, k/2) = 0.$$

We remark that the method of proof in [14] is mainly based on complex analytic arguments, while our proof of Theorem 8.1 crucially depends on *p*-adic methods. Now we explain the second application. We conditionally prove a congruence between *L*-values which is predicted by the non-commutative Iwasawa theory:-

THEOREM (See Theorem 9.1). *Let σ be the representation of Δ induced from the trivial character of Δ_0 . Let \mathfrak{p} be the maximal ideal of $\mathcal{I}(\mu_p)$. Assume that every prime divisor of *m* is inert or ramified in \mathcal{K} . Assuming Hypothesis 4, which is given in Section 9, we have*

$$\mathcal{L}_p(f, \rho, n) \equiv \mathcal{L}_p(f, \sigma, n) \pmod{\mathfrak{p}}. \tag{2.10}$$

We construct $H(\rho, T)$ by using the measure constructed in [10] and [13], and prove (2.9) and (2.10) by explicit calculation of the aforementioned measure.

3. Review of the Katz–Hida–Tilouine measure

In this section, we review the properties of measure constructed by Katz [13] and Hida–Tilouine [10]. We follow the notation of [10]. Put

$$\mathcal{F} = F\mathcal{K}, \text{ and } \mathcal{M} = M\mathcal{K},$$

and let $\mathbb{N}_{\mathcal{F}/\mathbb{Q}}$ denote the norm map from \mathcal{F} to \mathbb{Q} . For a number field *L*, We denote by L^{cyc} the cyclotomic \mathbb{Z}_p -extension of *L*. The restriction maps induce isomorphisms

$$\begin{aligned} \text{Gal}(\mathcal{M}/\mathcal{K}) &\longrightarrow \text{Gal}(M/\mathbb{Q}), \quad \text{Gal}(\mathcal{M}/\mathcal{F}) \longrightarrow \text{Gal}(M/F), \text{ and} \\ \text{Gal}(\mathcal{F}^{\text{cyc}}/\mathcal{F}) &\longrightarrow \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q}). \end{aligned}$$

Keeping the notation of the previous section, we denote by η a fixed non-trivial character of $\text{Gal}(M/F)$ and by χ a Dirichlet character of *p*-power order and conductor. Using the second and third isomorphisms, we may view η and χ as characters of $\text{Gal}(\mathcal{M}/\mathcal{F})$ and $\text{Gal}(\mathcal{F}^{\text{cyc}}/\mathcal{F})$, respectively. Via class field theory, we may then view η and χ as characters of $\mathbb{A}_{\mathcal{F}}^{\times}$, which we denote by $\eta_{\mathcal{F}}$ and $\chi_{\mathcal{F}}$. We also denote by $\mathbb{N}_{\mathcal{F}/\mathcal{K}}: \mathbb{A}_{\mathcal{F}}^{\times} \rightarrow \mathbb{A}_{\mathcal{K}}^{\times}$ the norm map and write $\bar{\psi}_{\mathcal{F}} = \bar{\psi} \circ \mathbb{N}_{\mathcal{F}/\mathcal{K}}$. If τ is any grossencharacter of A_0 -type, then let $\hat{\tau}$ be the *p*-adic avatar of τ . Also we write $\bar{\tau}$ for the grossencharacter defined by $\bar{\tau}(a) = c(\tau(a))$, where *c* is the complex conjugation. Let $N_{\mathcal{F}}$ be the norm grossencharacter $N_{\mathcal{F}}: \mathbb{A}_{\mathcal{F}} \rightarrow \mathbb{C}^{\times}$. For an integer *n*, define

$$\lambda = \eta_{\mathcal{F}} \chi_{\mathcal{F}} N_{\mathcal{F}}^{-n} \cdot \bar{\psi}_{\mathcal{F}}^{k-1}. \tag{3.1}$$

Recall that we denoted the ideal theoretic grossencharacter associated to τ by τ_f . Let \tilde{p} the prime of \mathcal{K} such that $\bar{\psi}_f^{k-1}(\tilde{p}\mathcal{O}_{\mathcal{K}}) = \alpha$, where we recall that α is the *p*-adic inverse root occurring in (2.7). If we define Σ to be the set of embeddings of \mathcal{F} into \mathbb{C} which extends the given embedding $i_{\infty}: \mathcal{K} \rightarrow \mathbb{C}$, and denote by \mathcal{F}^+ the maximal real subfield of \mathcal{F} , then Σ is a CM type for $\mathcal{F}/\mathcal{F}^+$ in the sense of [10] and [13]. We now briefly recall the properties of

Katz-Hida-Tilouine measure, using the notation of [10]. Let \mathfrak{c} be the exact conductor of λ . We decompose $\mathfrak{c} = \mathfrak{f}\mathfrak{i}$, where the prime divisors of \mathfrak{f} lie above the primes of \mathcal{F}^+ which split in \mathcal{F} and the prime divisors of \mathfrak{i} lie above the primes of \mathcal{F}^+ inert or ramified in \mathcal{F} . Depending on the choice of $\delta \in \mathcal{F}$ subject to some conditions, we have the p -adic period $\Omega_{\mathfrak{p}} \in \mathcal{I}^\times$ and the complex period $\Omega_\infty \in \mathcal{F}^+ \otimes_{\mathbb{Q}} \mathbb{C}$. Using the CM type Σ we have chosen, we may write $\Omega_\infty = (\Omega_{\infty,\sigma})_{\sigma \in \Sigma}$. We do not recall all of the conditions that δ must satisfy, but mention that it is possible to choose such a δ so that $\sigma(\delta)$ is totally imaginary with positive imaginary part for all $\sigma \in \Sigma$, and

$$\text{ord}_{\mathfrak{p}}(\delta\mathcal{O}_{\mathcal{F}}) = d \tag{3.2}$$

where \mathfrak{p} is the prime of \mathcal{F} corresponding to our fixed embedding $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_{\mathfrak{p}}$, d is the power of \mathfrak{p} occurring in the absolute different $\mathfrak{d}_{\mathcal{F}}$ of \mathcal{F} , and $\text{ord}_{\mathfrak{p}}(r)$ for an ideal r of $\mathcal{O}_{\mathcal{F}}$ is the exponent of \mathfrak{p} appearing in the decomposition of r into prime ideals of $\mathcal{O}_{\mathcal{F}}$. For the details regarding the choice of δ , see [10, (0.9)]. For definition of the local root number $W_{\mathfrak{p}}(\tau)$ we use for a grossencharacter τ , see [10, (0.8)]. We note that the authors of [10] often write Σ to denote the set of places Σ_p of \mathcal{F} induced by embeddings $i_p \circ i_\infty^{-1} \circ \sigma$ for $\sigma \in \Sigma$, but in our case Σ_p consists of single place \mathfrak{p} , and use \mathfrak{p} in our formulae. If $\xi = \sum_{\sigma \in \Sigma} (n_\sigma + m_\sigma c)\sigma$ is an element of free abelian group generated by $\Sigma \cup c\Sigma$, as in [10, theorem II], we define

$$\Gamma_\Sigma(\xi) = \prod_{\sigma} \Gamma(n_\sigma), \text{ and } x^\xi = \prod_{\sigma} x_\sigma^{n_\sigma} \prod_{\sigma} c(x_\sigma)^{m_\sigma}.$$

for $x = (x_\sigma)_{\sigma \in \Sigma} \in \mathbb{C}^\Sigma$. Let $D(K)$ denote the discriminant of K when K is a number field. Let $G_\infty(\mathfrak{c})$ be the projective limit of the ray class groups of \mathcal{F} of conductor $p^n \mathfrak{c}$ as n tends to infinity. For a grossencharacter λ of $\mathbb{A}_{\mathcal{F}}^\times$, define λ^* by $\lambda^*(x) = \lambda(x^c)^{-1} N_{\mathcal{F}}(x)$. We also put $\text{Im}(\delta) = (y_\sigma)_{\sigma \in \Sigma} \in \mathbb{C}^\Sigma$ where y_σ denotes the imaginary part of $\sigma(\delta)$. Let \mathfrak{c} be an ideal of \mathcal{F} prime to p and decompose $\mathfrak{c} = \mathfrak{f}\mathfrak{i}$ so that $\mathfrak{f}\mathfrak{i}$ consists of primes split over \mathcal{F}^+ , \mathfrak{i} consists of inert or ramified primes over \mathcal{F}^+ , $\mathfrak{f} + \mathfrak{i} = \mathcal{O}_{\mathcal{F}}$, and $(\mathfrak{i})^c \supset \mathfrak{f}$. Finally, we put $w_{\mathcal{F}} = [\mathcal{O}_{\mathcal{F}}^\times : \mathcal{O}_{\mathcal{F}^+}^\times]$, which is consistent with (2.1) if we replace \mathcal{F} by \mathcal{K} . Note that $\text{ord}_p(w_{\mathcal{F}}\mathbb{Z}) = 1$.

THEOREM 3.1 (Katz, Hida–Tilouine). *There exists a unique measure μ on $G_\infty(\mathfrak{c})$ having values in \mathcal{I} satisfying*

$$\begin{aligned} \frac{\int_{G_\infty(\mathfrak{c})} \hat{\lambda} d\mu}{\Omega_{\mathfrak{p}}^{(p-1)m_0 + \sum_{\sigma} 2d_\sigma}} &= w_{\mathcal{F}} W_{\mathfrak{p}}(\tau) \frac{(-1)^{m_0 \Sigma} \pi^d \Gamma_\Sigma(m_0 \Sigma + d)}{\sqrt{|D(\mathcal{F}^+)|} |\text{Im}(\delta)|^d \Omega_\infty^{m_0 \Sigma + 2d}} \prod_{\mathfrak{t} | \mathfrak{c}} (1 - \lambda(\mathfrak{t})) \\ &\times \prod_{\mathfrak{p} \in \Sigma} ((1 - \lambda(\mathfrak{p}^c))(1 - \lambda^*(\mathfrak{p}^c))) L(0, \tau) \end{aligned}$$

for all grossencharacters λ with conductor dividing $\mathfrak{c}p^\infty$ such that:

- (i) the conductor of λ is divisible by all prime factors of \mathfrak{f} ;
- (ii) the infinity type of λ is $m_0 \Sigma + d(1 - c)$ for some integers m_0, d_σ , and $d = \sum_{\sigma \in \Sigma} d_\sigma \sigma$, which satisfy either $m_0 \geq 1$ and $d_\sigma \geq 0$ or $m_0 \leq 1$ and $d_\sigma \geq 1 - m_0$.

Here we choose the normalization of the infinity type as

$$\lambda(x_\infty) = \prod_{\sigma \in \Sigma} (x_\sigma)^{m_0} \left(\frac{x_\sigma}{x_\sigma^c} \right)^{d_\sigma}.$$

The infinity type of $N_{\mathcal{F}}^{-n} \cdot \bar{\psi}_{\mathcal{F}}^{k-1}$ is

$$N_{\mathcal{F}}^{-n} \bar{\psi}_{\mathcal{F}}^{k-1}(x_{\infty}) = \prod_{\sigma \in \Sigma} (x_{\sigma})^{-k+1+2n} \left(\frac{x_{\sigma}}{x_{\sigma}^c} \right)^{k-1-n}.$$

Therefore, the interpolation range is

$$\begin{aligned} -k + 1 + 2n &\geq 1 \text{ and } k - 1 - n \geq 0, \text{ or} \\ -k + 1 + 2n &\leq 1 \text{ and } n - 1 \geq 0. \end{aligned}$$

In other words, the condition in the theorem is satisfied precisely when $n = 1, 2, \dots, k - 1$. To see the *L*-values interpolated, note that

$$\begin{aligned} L(s, \kappa^{-n} \tau) &= \prod_{\mathfrak{r}} (1 - \tau(\mathfrak{r}) N_{\mathcal{F}}(\mathfrak{r})^{-n} N_{\mathcal{F}}(\mathfrak{r})^{-s})^{-1} \\ &= L(n + s, \tau). \end{aligned}$$

Therefore

$$L(0, \kappa^{-n} \tau) = L(n, \tau).$$

The next three sections will be to compare various restrictions of Katz–Hida–Tilouine measure to the cyclotomic line with the *L*-functions discussed in Section 2.

4. Computation of Euler factors

The aim of this section is to show that the modified Euler factors at the finite primes occurring in (2.8) are precisely the same as those coming from the Katz–Hida–Tilouine measure. We first prove:

PROPOSITION 4.1. *Let r be any rational prime and let $S_r(\mathcal{F})$ be the set of places of \mathcal{F} lying above r . Let η' be either η or the trivial character, and θ be ρ or σ respectively. Then, for any Dirichlet character χ of p -power conductor and order, we have*

$$\prod_{\mathfrak{r} \in S_r(\mathcal{F})} P_{\mathfrak{r}}(\chi \eta' \bar{\psi}_{\mathcal{F}}^{k-1}, N_{\mathcal{F}}(\mathfrak{r})^{-s}) = P_r(f, \theta \chi, r^{-s}).$$

Proof. To lighten the notation, let ν be $\chi \eta' \bar{\psi}_{\mathcal{F}}^{k-1}$. Let ℓ be a rational prime and choose a prime \mathfrak{l} of \mathcal{F} lying above ℓ . Let $\hat{\nu}^{\mathfrak{l}}$ be the \mathfrak{l} -adic avatar of ν and denote by $V_{\mathfrak{l}}(\nu)$ the associated one dimensional vector space over $\overline{\mathbb{Q}}_{\ell}$ on which $\text{Gal}(\overline{\mathbb{Q}}/\mathcal{F})$ acts by $\hat{\nu}^{\mathfrak{l}}$. Then, it is easy to see that $\text{Ind} V_{\mathfrak{l}}(\nu)$ is isomorphic to $V_{\ell}(f) \otimes_{\overline{\mathbb{Q}}_{\ell}} V_{\ell}(\theta \chi)$, where $\text{Ind} V_{\mathfrak{l}}(\nu)$ is the induction of $V_{\mathfrak{l}}(\nu)$ to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Therefore the assertion follows from Artin formalism of Euler factors.

We also prove the next lemma.

LEMMA 4.2. *Recall that we defined λ by (3.1), and λ^* by $\lambda^*(x) = \lambda(x^c)^{-1} N_{\mathcal{F}}(x)$. Then we have*

$$P_p(f, \rho \chi, p^{-n}) \frac{P_p(\widehat{\rho \chi}, \frac{p^{n-1}}{\alpha})}{P_p(\rho \chi, \frac{\alpha}{p^n})} = (1 - \lambda(\mathfrak{p}^c)) (1 - \lambda^*(\mathfrak{p}^c)).$$

Proof. First assume that χ is nontrivial. In that case, λ is ramified at all places of \mathcal{F} above p . On the other hand, $V_{\ell}(\rho \chi)^{f_p} = 0$ since Γ acts trivially on $V_{\ell}(\rho)$ and therefore it

acts via χ on $V_\ell(\rho\chi)$. Thus both sides are equal to 1. On the other hand, let χ be trivial. We consider two subcases; the unique prime \wp of F lying above p is ramified in M or not. Suppose \wp is ramified. λ is ramified at \mathfrak{p} since η is ramified at \mathfrak{p} . On the other hand, we have $I_p(\Delta) = \Delta$ and its action on $V(\rho)$ has no invariant vectors. Thus, again, both sides are equal to 1. Now we consider the case when \wp is unramified in M . Then \wp splits completely or inert in M . In any case, we have $I_p(\Delta) = \text{Gal}(M_{\mathfrak{p}}/\mathbb{Q}_p(m^{1/p})) \cong \text{Gal}(F/\mathbb{Q})$, so $V_\ell(\rho)^{I_p} = \overline{\mathbb{Q}}_\ell \cdot (e_1 + \cdots + e_{p-1})$ on which Frob_p acts trivially, or by multiplication by $1 = \eta(\text{Frob}_\wp)$. We need to show

$$\begin{aligned} P_p(f, \rho, p^{-n}) \frac{1 - \frac{p^{n-1}}{\alpha}}{1 - \frac{\alpha}{p^n}} &= (1 - p^{-n}\alpha)(1 - \kappa^{n-1}(\tilde{p})\tilde{\psi}^{-1}(\tilde{p})) \\ &= (1 - p^{-n}\beta)(1 - p^{-n}\alpha). \end{aligned}$$

It easily follows from

$$P_p(f, X) = (1 - \alpha X)(1 - \beta X).$$

5. Comparison of periods

In this section we compare three kinds of periods attached to f . The first one is the classical period defined in (2.2). The second one is the Katz–Hida–Tilouine period over \mathcal{K} , and the third is the Katz–Hida–Tilouine period over \mathcal{F} . Also, we will write $\Omega_\infty^{\text{KHT}}(K)$ to denote the period Ω_∞ attached to the field K in Section 3, where K is \mathcal{F} or \mathcal{K} . Recall that, in Section 3, we fixed $\delta \in \mathcal{F}$, for the CM extension $\mathcal{F}/\mathcal{F}^+$. We emphasize that δ is attached to $\mathcal{F}/\mathcal{F}^+$ by writing $\delta = \delta_{\mathcal{F}}$, and fix $\delta_{\mathcal{K}} \in \mathcal{K}$ corresponding to \mathcal{K}/\mathbb{Q} satisfying [10, (0.9)]. Note that $\delta_{\mathcal{K}}$ is a p -adic unit. We will use the notation of [10], especially Section 5.1. In particular, A will denote the valuation ring of $\overline{\mathbb{Q}}$ with respect to i_p .

PROPOSITION 5.1. *If we write*

$$\Omega_\infty^{\text{KHT}}(\mathcal{F}) = (\Omega_{\infty, \sigma}^{\text{KHT}}(\mathcal{F}))_{\sigma \in \Sigma},$$

then we have

$$\Omega_{\infty, \sigma}^{\text{KHT}}(\mathcal{F}) = (2\delta_{\mathcal{K}})\sigma(2\delta_{\mathcal{F}})^{-1}\Omega_\infty^{\text{KHT}}(\mathcal{K}).$$

Proof. Following [13, section 5.1], let $X(\mathcal{O}_{\mathcal{F}})$ (resp. $X(\mathcal{O}_{\mathcal{K}})$) be the Hilbert–Blumenthal abelian variety over A corresponding to the ideal $\mathcal{O}_{\mathcal{F}}$ (resp. $\mathcal{O}_{\mathcal{K}}$). Let us choose a non-vanishing differential $\omega_{\mathcal{F}}$ on $X(\mathcal{O}_{\mathcal{F}})/A$ in the sense of [13, section (1.0.3)]. Similarly, we choose a non-vanishing differential $\omega_{\mathcal{K}}$ on $X(\mathcal{O}_{\mathcal{K}})$. By definition, they give isomorphisms

$$\omega_{\mathcal{F}}: \text{Lie}(X(\mathcal{O}_{\mathcal{F}})) \longrightarrow \mathfrak{d}_{\mathcal{F}}^{-1} \otimes_{\mathbb{Z}} A, \text{ and } \omega_{\mathcal{K}}: \text{Lie}(X(\mathcal{O}_{\mathcal{K}})) \longrightarrow \mathfrak{d}_{\mathcal{K}}^{-1} \otimes_{\mathbb{Z}} A,$$

where \mathfrak{d}_K denotes the different ideal for a number field K . Since p splits in \mathcal{K} , we have canonical isomorphisms

$$2\delta_{\mathcal{K}}^{-1} \cdot A \oplus 2\delta_{\mathcal{K}}^{-c} \cdot A \cong \mathfrak{d}_{\mathcal{K}}^{-1} \otimes_{\mathbb{Z}} A, \text{ and } \bigoplus_{\sigma \in \Sigma} (\sigma(2\delta_{\mathcal{F}})^{-1} \cdot A \oplus \sigma(2\delta_{\mathcal{F}})^{-c} \cdot A) \cong \mathfrak{d}_{\mathcal{F}}^{-1} \otimes_{\mathbb{Z}} A. \tag{5.1}$$

As abelian varieties over A , $X(\mathcal{O}_{\mathcal{F}})$ is isomorphic to $X(\mathcal{O}_{\mathcal{K}}) \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{O}_{\mathcal{F}}$. In particular, as complex analytic abelian varieties, $X(\mathcal{O}_{\mathcal{K}})(\mathbb{C}) \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{O}_{\mathcal{F}}$ is canonically isomorphic to

$X(\mathcal{O}_{\mathcal{F}})(\mathbb{C})$. Now we recall the definition of the nowhere vanishing differential $\omega_{\text{trans},\mathcal{F}}$ on $X(\mathcal{O}_{\mathcal{F}})(\mathbb{C})$. We write u for the standard coordinate for \mathbb{C}^{Σ} and denote by du the associated differential on $\mathbb{C}^{\Sigma}/\mathcal{O}_{\mathcal{F}}$. We define $\omega_{\text{trans},\mathcal{F}}$ by the pull-back of du under the isomorphism $X(\mathcal{O}_{\mathcal{F}})(\mathbb{C}) \rightarrow \mathbb{C}^{\Sigma}/\mathcal{O}_{\mathcal{F}}$. We similarly define a nowhere vanishing differential $\omega_{\text{trans},\mathcal{K}}$ on $X(\mathcal{O}_{\mathcal{K}})(\mathbb{C})$. The canonical isomorphism between $X(\mathcal{O}_{\mathcal{F}})$ and $X(\mathcal{O}_{\mathcal{K}}) \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{O}_{\mathcal{F}}$ identifies $\omega_{\text{trans},\mathcal{F}}$ on $X(\mathcal{O}_{\mathcal{F}})(\mathbb{C})$ with $\omega_{\text{trans},\mathcal{K}} \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{O}_{\mathcal{F}}$ on $X(\mathcal{O}_{\mathcal{F}})(\mathbb{C})$ via the pull-back. Now the assertion of Proposition follows from definition of periods; $\Omega_{\infty}^{\text{KHT}}(K)\omega_{\text{trans},K} = \omega_K$ when K is \mathcal{K} or \mathcal{F} .

PROPOSITION 5.2. *Recall that d is defined to be the power of \mathfrak{p} in $\mathfrak{d}_{\mathcal{F}}$. We have*

$$\frac{\pi^{\sum(k-1-n)\sigma}}{\Omega_{\infty}^{\text{KHT}}(\mathcal{F})^{(k-1)\sum\sigma} \cdot \prod_{\sigma \in \Sigma} \text{Im}(\sigma(\delta_{\mathcal{F}}))^{k-1-n}} = \frac{v \cdot u^n}{(2\pi i)^{n(p-1)} p^{-nd} (\Omega_E^+ \Omega_E^-)^{(p-1)(k-1)/2}}$$

where u and v are non-zero elements of \mathcal{K} prime to \tilde{p} .

Proof. Let $\delta_0 := \prod_{\sigma \in \Sigma} \sigma(2\delta_{\mathcal{F}})$ and $\delta'_0 = (2\delta_{\mathcal{K}})^{p-1}$. Note that $\delta_0 \in \mathcal{K}$. Let

$$u_{\pm} = \frac{\Omega_{\infty}^{\text{KHT}}(\mathcal{K})}{2\pi i \cdot \Omega_E^{\pm}}.$$

Then we have $u_{\pm} \in \mathbb{Z}_p^{\times}$. Let $u_0 = (u_+ u_-)^{(k-1)(p-1)/2}$. If we let

$$u = p^d / \delta_0, \tag{5.2}$$

then u is in \mathcal{K} and prime to \tilde{p} , since \mathcal{F}/\mathcal{K} is totally ramified at \tilde{p} and \mathcal{K}/\mathbb{Q} is unramified at p . We have

$$\Omega_{\infty}^{\text{KHT}}(\mathcal{F})^{(k-1)\sum\sigma} \prod_{\sigma \in \Sigma} \text{Im}(\sigma(\delta_{\mathcal{F}}))^{k-1-n} \tag{5.3}$$

$$= \left(\prod_{\sigma} (2\delta_{\mathcal{K}})\sigma(2\delta_{\mathcal{F}})^{-1} \Omega_{\infty}^{\text{KHT}}(\mathcal{K}) \right)^{k-1} \prod_{\sigma} \text{Im}(\sigma(\delta_{\mathcal{F}}))^{k-1-n} \tag{5.4}$$

$$= \left(\prod_{\sigma} (2\delta_{\mathcal{K}})\sigma(2\delta_{\mathcal{F}})^{-1} (2\pi i) \right)^{k-1} (\Omega_E^+ \Omega_E^-)^{(k-1)(p-1)/2} u_0 \prod_{\sigma} \text{Im}(\sigma(\delta_{\mathcal{F}}))^{k-1-n} \tag{5.5}$$

$$= u_0 (\delta'_0 \delta_0^{-1})^{k-1} (\Omega_E^+ \Omega_E^-)^{(k-1)(p-1)/2} (2\pi i)^{n(p-1)} (2\pi i)^{(k-1-n)(p-1)} \prod_{\sigma} \text{Im}(\sigma(\delta_{\mathcal{F}}))^{k-1-n} \tag{5.6}$$

$$= u_0 (\delta'_0 \delta_0^{-1})^{k-1} (\Omega_E^+ \Omega_E^-)^{(k-1)(p-1)/2} (2\pi i)^{n(p-1)} \pi^{(k-1-n)(p-1)} \delta_0^{k-1-n} \tag{5.7}$$

$$= u_0 (\delta'_0)^{k-1} (\Omega_E^+ \Omega_E^-)^{(k-1)(p-1)/2} (2\pi i)^{n(p-1)} \pi^{(k-1-n)(p-1)} \delta_0^{-n} \tag{5.8}$$

$$= v (\Omega_E^+ \Omega_E^-)^{(k-1)(p-1)/2} (2\pi i)^{n(p-1)} \pi^{(k-1-n)(p-1)} p^{-nd} u^n \tag{5.9}$$

by letting $v = u_0 (\delta'_0)^{k-1}$. Note that by one of the defining properties of $\delta_{\mathcal{F}}$, the imaginary part of $\sigma(\delta_{\mathcal{F}})$ is positive for all $\sigma \in \Sigma$, whence (5.6) equals (5.7).

6. Comparison of epsilon factors

It is important for our purpose to compute the discrepancy between the local factors in [10, (0.10)] and the epsilon factors of Tate–Deligne. We recall the definition of the canonical additive character $\Psi = \Psi_{\mathbb{Q}}: \mathbb{A}_{\mathbb{Q}} \rightarrow \mathbb{C}$. For $v = \infty$, define $\Psi_{\infty}(x_{\infty}) = \exp(2\pi i x_{\infty})$. For a finite place $v = \ell$, the natural injection from \mathbb{Q} to \mathbb{Q}_{ℓ} identifies the ℓ -primary subgroup of \mathbb{Q}/\mathbb{Z} with $\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}$, so we regard an element $x_{\ell} \in \mathbb{Q}_{\ell}$ as an element in \mathbb{Q}/\mathbb{Z} , for the

moment. We define $\Psi_\ell(x_\ell) = \exp(-x_\ell)$. Then for an element $x = (x_v)_v \in \mathbb{A}_\mathbb{Q}$, we define $\Psi(x) = \prod_v \Psi_v(x_v)$, where v runs over the set of all places of \mathbb{Q} . The product is well defined since $\Psi_\ell(x_\ell) = 1$ for $x_\ell \in \mathbb{Z}_\ell$, and Ψ is trivial on \mathbb{Q} . For a number field K , we denote by $\Psi_K: \mathbb{A}_K \rightarrow \mathbb{C}$ the canonical additive character defined by $\Psi_K := \Psi_\mathbb{Q} \circ \text{Tr}_{K/\mathbb{Q}}$, where $\text{Tr}_{K/\mathbb{Q}}$ denotes the trace map. We denote by dx a Haar measure on ideles normalized so that the group of local ring of integers has volume 1. Recall that $\mathfrak{d}_\mathcal{F}$ is the absolute different of \mathcal{F} , and we let $\mathfrak{d}_{\mathcal{F}_\mathfrak{p}}$ denote the \mathfrak{p} -component of $\mathfrak{d}_\mathcal{F}$. For a prime ideal P of a number field K and an ideal r of \mathcal{O}_K , we denote by $\text{ord}_P(r)$ the exponent of P appearing in the decomposition of r into prime ideals of \mathcal{O}_K . The discrepancy that we want to compute is given in the following lemma.

LEMMA 6.1 ([1, lemma 3.3]). *For a grossencharacter τ of $\mathbb{A}_\mathcal{F}^\times$, we have*

$$\epsilon_{\mathfrak{p}}(\tau^{-1}, \Psi_{\mathcal{F}}, dx) = N(\mathfrak{p})^{\text{ord}_{\mathfrak{p}}(f(\tau))} \tau_{\mathfrak{p}}^{-1}(-2\delta_{\mathcal{F}}) N(\mathfrak{d}_{\mathcal{F}_\mathfrak{p}}) W_{\mathfrak{p}}(\tau), \tag{6.1}$$

where $W_{\mathfrak{p}}(\tau)$ is the local root number given by [10, (0.10)].

We want to remind the reader that the author [1] abuses notation in a way that $\tau_{\mathfrak{p}}(\delta_{\mathcal{F}})$ in [1, lemma 3.1] actually means $\tau_{\mathfrak{p}}(-2\delta_{\mathcal{F}})$. We do not follow his convention and use the honest $\tau_{\mathfrak{p}}(-2\delta_{\mathcal{F}})$. After a routine computation, we conclude:

LEMMA 6.2. *Let η' denote either η or the trivial character $\mathbf{1}$, and θ be its induced character of $\text{Gal}(M/\mathbb{Q})$. If we let $\lambda' = \eta' \chi \bar{\psi}_{\mathcal{F}}^{k-1} N_{\mathcal{F}}^{-n}$, we have*

$$W_{\mathfrak{p}}(\lambda') = \epsilon_{\mathfrak{p}}(\theta \chi, \Psi, dx) \left(\frac{p^{n-1}}{\alpha} \right)^{e_{\mathfrak{p}}(\theta \chi)} \alpha^d p^{-nd} \eta' \chi_{\mathfrak{p}}(-2\delta_{\mathcal{F}}). \tag{6.2}$$

Proof. Equation (6.1) can be rewritten as

$$W_{\mathfrak{p}}(\lambda') = \epsilon_{\mathfrak{p}}(\lambda^{-1}, \Psi_{\mathcal{F}}, dx) N(\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(f(\lambda'))} \lambda'_{\mathfrak{p}}(-2\delta_{\mathcal{F}}) N(\mathfrak{d}_{\mathcal{F}_\mathfrak{p}})^{-1}. \tag{6.3}$$

Since $\psi_{\mathcal{F}}$ and $N_{\mathcal{F}}$ are unramified at \mathfrak{p} , we have $\text{ord}(f(\lambda')) = \text{ord}(f(\eta' \chi))$. Denote by $\varpi_{\mathfrak{p}} \in \mathcal{F}_{\mathfrak{p}}$ the uniformizer of \mathfrak{p} . By the conductor-discriminant formula, $f(\eta' \chi) N_{\mathcal{F}}(\mathfrak{d}_{\mathcal{F}_\mathfrak{p}}) = f(\theta \chi)$, hence Equation (6.3) continues as

$$\begin{aligned} W_{\mathfrak{p}}(\lambda') &= \epsilon_{\mathfrak{p}}((\eta' \chi)^{-1} \bar{\psi}_{\mathcal{F}}^{1-k} N_{\mathcal{F}}^n, \Psi_{\mathcal{F}}, dx) N(\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(f(\eta' \chi))} \lambda'_{\mathfrak{p}}(-2\delta_{\mathcal{F}}) N(\mathfrak{d}_{\mathcal{F}_\mathfrak{p}})^{-1} \\ &= \epsilon_{\mathfrak{p}}((\eta' \chi)^{-1} \bar{\psi}_{\mathcal{F}}^{1-k} N_{\mathcal{F}}^n, \Psi_{\mathcal{F}}, dx) p^{-e_{\mathfrak{p}}(\theta \chi)} \lambda'_{\mathfrak{p}}(-2\delta_{\mathcal{F}}). \end{aligned}$$

Recall that d is the power of \mathfrak{p} in $\mathfrak{d}_{\mathcal{F}}$. Then the level of $\Psi_{\mathcal{F}}$, which we denote by $n(\Psi_{\mathcal{F}})$, equals d . Also, we recall from (3.2) that $\text{ord}_{\mathfrak{p}}(\delta_{\mathcal{F}}) = d$. Using the standard formula

$$\epsilon_{\mathfrak{p}}((\eta' \chi)^{-1} \bar{\psi}_{\mathcal{F}}^{1-k} N_{\mathcal{F}}^n, \Psi_{\mathcal{F}}, dx) = \epsilon_{\mathfrak{p}}((\eta' \chi)^{-1}, \Psi_{\mathcal{F}}, dx) \bar{\psi}_{\mathcal{F}}^{1-k} N_{\mathcal{F}}^n \left(\varpi_{\mathfrak{p}}^{n(\Psi_{\mathcal{F}}) + \text{ord}_{\mathfrak{p}}(f(\eta' \chi))} \right),$$

we conclude

$$\begin{aligned} W_{\mathfrak{p}}(\lambda') &= \epsilon_{\mathfrak{p}}((\eta' \chi)^{-1}, \Psi_{\mathcal{F}}, dx) \bar{\psi}_{\mathcal{F}}^{1-k} N_{\mathcal{F}}^n \left(\varpi_{\mathfrak{p}}^{n(\Psi_{\mathcal{F}}) + \text{ord}_{\mathfrak{p}}(f(\eta' \chi))} \right) p^{-e_{\mathfrak{p}}(\theta \chi)} \lambda'_{\mathfrak{p}}(-2\delta_{\mathcal{F}}) \\ &= \epsilon_{\mathfrak{p}}((\eta' \chi)^{-1}, \Psi_{\mathcal{F}}, dx) \left(\frac{p^{n-1}}{\alpha} \right)^{e_{\mathfrak{p}}(\theta \chi)} \lambda'_{\mathfrak{p}}(-2\delta_{\mathcal{F}}) \\ &= \epsilon_{\mathfrak{p}}((\eta' \chi)^{-1}, \Psi_{\mathcal{F}}, dx) \left(\frac{p^{n-1}}{\alpha} \right)^{e_{\mathfrak{p}}(\theta \chi)} \alpha^d p^{-nd} (\eta' \chi)_{\mathfrak{p}}(-2\delta_{\mathcal{F}}). \end{aligned}$$

LEMMA 6.3. *Keeping the notations of Lemma 6.2, we have*

$$\epsilon_p(\theta\chi, \Psi_{\mathbb{Q}}, dx) = \epsilon_p(\eta'\chi, \Psi_{\mathcal{F}}, dx) \cdot \sqrt{|D(F)|}.$$

Proof. Since epsilon factors are inductive in virtual representations of dimension zero, we have

$$\frac{\epsilon_{\tilde{p}}(\theta\chi, \Psi_{\mathbb{Q}}, dx)}{\epsilon_{\tilde{p}}(\sigma, \Psi_{\mathbb{Q}}, dx)} = \frac{\epsilon_p(\eta, \Psi_{\mathcal{F}}, dx)}{\epsilon_p(\mathbf{1}, \Psi_{\mathcal{F}}, dx)}.$$

Let \wp be the unique prime of F lying above p . Then $\mathcal{F}_{\wp} \cong F_{\wp}$, and $\mathcal{K}_{\tilde{p}} \cong \mathbb{Q}_{\tilde{p}}$. Therefore, the assertion of the lemma follows from the conductor-discriminant formula for F/\mathbb{Q} .

7. Proof of the main theorem

We begin with a lemma.

LEMMA 7.1. *There exists a power series $H_u(T) \in \mathbb{Z}_p[[T]]$ such that $H_u((1+p)^n - 1) = u^n$. Furthermore, $H_u(T) \in \mathbb{Z}_p[[T]]^{\times}$.*

Proof. The second assertion is obvious once we have existence of such $H_u(T)$. Recall that u is defined in (5.2). By local class field theory, we know that u actually lies in $1 + p\mathbb{Z}_p$. We obtain such an $H_u(T)$ by letting $H_u(T) = (1 + T)^a$ with $a = \log_p u / \log_p(1 + p)$.

We also introduce the following hypothesis, which is forced upon us by properties of the Katz–Hida–Tilouine measure, although it is conjecturally not at all necessary for the p -adic L -functions discussed in Section 2.

HYPOTHESIS 3. *Every prime divisor of m is ramified or inert in \mathcal{K} .*

We identify $\mathcal{I}[[T]] \cong \Lambda(\Gamma)$ and $\Gamma \cong 1 + \mathbb{Z}_p$. In particular we view a Dirichlet character of p -power order and conductor as a character of Γ . Now we are ready to prove the main theorem.

THEOREM 7.2. *Suppose Hypotheses 1, 2 and 3 hold. Let θ denotes ρ or σ . Then, there exists $H(\sigma, T)$ in $\mathcal{I}[[T]]/p$ and $H(\rho, T)$ in $\mathcal{I}(\mu_p)[[T]]/p$ such that for all characters of finite order character χ of Γ and integers $n = 1, 2, \dots, k - 1$, we have*

$$\frac{H(\theta, (1+p)^{-n}\chi(1+p) - 1)}{\Omega_p^{(p-1)(k-1)}} = \mathcal{L}_p(f, \theta\chi, n). \tag{7.1}$$

Proof. Let η' denote either η or the trivial character $\mathbf{1}$, and θ be its induced character of $\text{Gal}(M/\mathbb{Q})$. Let $\mu(\eta')$ be the specialization of the Katz–Hida–Tilouine measure μ to Γ via $\eta'\psi_{\mathcal{F}}^{k-1}$. More precisely, this specialization $\mu(\eta')$ is the unique element of $\Lambda(\Gamma)$ such that

$$\int_{\Gamma} \tau d\mu(\eta') = \int_{G_{\infty}(\mathfrak{c})} \eta'\psi_{\mathcal{F}}^{k-1} \tau d\mu \tag{7.2}$$

for all characters τ of Γ . Also, there is a Dirac-delta measure $\mu(-2\delta_{\mathcal{F}})$ on $G_{\infty}(\mathfrak{c})$ such that

$$\int_{G_{\infty}(\mathfrak{c})} \tau d\mu(-2\delta_{\mathcal{F}}) = \tau_p(-2\delta_{\mathcal{F}}).$$

In other words, $\mu(-2\delta_{\mathcal{F}})$ is the Dirac-delta measure associated to the image of $-2\delta_{\mathcal{F}}$ under the map

$$\mathcal{F}^\times \longrightarrow \mathcal{F}_p^\times \longrightarrow G_\infty(\mathfrak{c}),$$

where the the first map is natural inclusion and the second is the local Artin map. Here \mathfrak{c} denotes the prime to p conductor of λ . Now we switch to the convention of viewing $\mathcal{I}[[\Gamma]]$ as a power series ring. By Lemma 7.1, there is an element $\tilde{u} = H_u(T) \in \mathcal{I}[[T]]$ such that $H_u((1 + p)^n - 1) = u^n$. Note that $\text{ord}_p(w_{\mathcal{F}}\mathbb{Z}) = 1$, and $\sqrt{|D(F)|}/\sqrt{|D(\mathcal{F}^+)|}$ is a p -adic unit. If we let

$$\mu(f, \theta) := \left(\alpha^d v \tilde{u} w_{\mathcal{F}} \frac{\sqrt{|D(F)|}}{\sqrt{|D(\mathcal{F}^+)|}} \right)^{-1} \mu(\eta'),$$

then the power series $H(\theta, T)$ corresponding to $\mu(f, \theta)$ lies in $\frac{1}{p}\mathcal{I}[[T]]$. From Proposition 4.1, we have

$$L(f, \theta\chi, n) = L(\lambda', 0),$$

and the interpolation property of $H(\theta, T)$ is a direct consequence of Theorem 3.1, Lemma 6.2, Lemma 6.3 and Lemma 7.1. Note that in order to obtain the interpolation for the case $\eta' = \mathbf{1}$ where we integrate an imprimitive character, we need Hypothesis 3.

We remark that the series $H(\sigma, T)$ recovers the product of $p - 1$ branches of p -adic L -functions associated to characters of $\text{Gal}(F/\mathbb{Q})$, which were first constructed by Manin. We also give a remark for a reader who might wonder why we need Hypothesis 3. Let \mathfrak{c}' be the prime to p conductor of λ' . We could use the measure on the group $G_\infty(\mathfrak{c}')$, instead of that on $G_\infty(\mathfrak{c})$. It works fine and both power series $H(\rho, T)$ and $H(\sigma, T)$ exists with the interpolation property (7.1) under Hypotheses 1, and 2. However, they are not relevant for our application, namely Theorem 9.1. We will heavily rely on the fact that $\mu(\eta)$ and $\mu(\mathbf{1})$ is are restrictions of a single measure on the same group $G_\infty(\mathfrak{c})$.

8. A generalization of Rohrlich's theorem

The author has been inspired by a comment in Mathoverflow made by a mysterious user named Jupiter, which suggested to him to prove the following theorem. It seems to be the first generalization of an important theorem of Rohrlich in [14], from the classical L -functions to a twist of a classical L -function by a non-abelian Artin representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The main theorem in [14], although it allows χ to vary among a bigger family of characters, can be viewed as the case when ρ is trivial.

THEOREM 8.1. *Let us assume that $k \geq 4$. As χ varies among all characters of Γ of finite order, there are only finitely many χ with*

$$L(f, \rho\chi, k/2) = 0.$$

Proof. By the Weierstrass preparation theorem, a non-zero power series in $\mathcal{I}[[T]]$ can have only finitely many zeros. Therefore it suffices to show that $H(\rho, T) \neq 0$. Note that the Euler product for $L(f, \rho, s)$ converges for s with $\text{Re}(s) > k/2 + 1/2$. In particular, $L(f, \rho\chi, k - 1) \neq 0$ if $k \geq 4$. Therefore, once we assume $k \geq 4$, it is guaranteed that $L(f, \rho, k - 1) \neq 0$. Since $H(\rho, T)$ interpolates the value $L(f, \rho\chi, k - 1)$, we conclude $H(\rho, T) \neq 0$.

9. Congruences predicted by non-commutative Iwasawa theory

Unfortunately, our next application depends on the following hypothesis.

HYPOTHESIS 4. *The Katz–Hida–Tilouine measure μ in Theorem 3.1 takes values in $p\mathcal{I}$.*

Even though Hypothesis 4 cannot be verified at this stage due to the limitation of author’s knowledge, the numerical data given in the next section strongly suggests that $H(\rho, T)$ is divisible by p . The author hopes to pursue this problem in the near future, following a suggestion of Professor. Hida. He proved in [9, theorem 5.1], a formula for the μ -invariants of branches of Katz–Hida–Tilouine measure when p is unramified in the relevant totally real field, which is not the case of ours since p is ramified in \mathcal{F}^+ . Indeed, his formula for the μ -invariant, namely [9, (5.27)], suggests that the μ -invariants of each branch of Katz–Hida–Tilouine measure is positive when p is ramified in the relevant totally real field, which is the case since p is ramified in \mathcal{F}^+ .

THEOREM 9.1. *Let \mathfrak{p} be the maximal ideal of $\mathcal{I}(\mu_p)$. Suppose that Hypotheses 1, 2, 3 and 4 hold. We have*

$$H(\rho, T) \equiv H(\sigma, T) \pmod{\mathfrak{p}\mathcal{I}(\mu_p)[[T]]}. \tag{9.1}$$

In particular, we have

$$\mathcal{L}_p(f, \rho, n) \equiv \mathcal{L}_p(f, \sigma, n) \pmod{p} \tag{9.2}$$

for $n = 1, 2, \dots, k - 1$.

Proof. Note that Hypothesis 4 implies that $H(\rho, T)$ and $H(\sigma, T)$ are contained in $\mathcal{I}[[T]]$ instead of $\frac{1}{p}\mathcal{I}[[T]]$. Then we have

$$\eta \equiv \eta^p = \mathbf{1} \pmod{\mathfrak{p}}.$$

Therefore we have $\mu(\eta) \equiv \mu(\mathbf{1})$ modulo $\mathfrak{p}\mathcal{I}(\mu_p)[[T]]$, where we recall that $\mu(\eta)$ and $\mu(\mathbf{1})$ were defined in (7.2) as the restrictions of the Katz–Hida–Tilouine measure, whence (9.1) follows. Note that the Hypothesis 4 is stronger than the positivity of the μ -invariants of both $H(\rho, T)$ and $H(\sigma, T)$. It is crucial for the proof of (9.1) that the entire measure takes value in $p\mathcal{I}$. Since both $\mathcal{L}_p(f, \rho, n)$ and $\mathcal{L}_p(f, \sigma, n)$ are rational numbers, (9.2) follows from (9.1).

We give an example which shows that the positivity of μ -invariants of both $H(\rho, T)$ and $H(\sigma, T)$ is not sufficient for Theorem 9.1. Recall that we wrote Δ_0 for the Galois group $\text{Gal}(\mathcal{M}/\mathcal{F})$, which is a cyclic group of order p . Let $H(T) \in \mathcal{I}[[T]]$ be a power series, and we consider

$$G = \sum_{g \in \Delta_0} gH(T) \in \mathbb{Z}_p[\Delta_0][[T]].$$

We view G as a measure on $\Delta_0 \times \Gamma$. Then, 0 is the largest integer r such that G takes value in $p^r\mathcal{I}$. If we take the branch of G , namely

$$G_\eta := \sum_{g \in \Delta_0} \eta(g)H(T)$$

for a character $\eta: \Delta_0 \rightarrow \overline{\mathbb{Q}}_p^\times$, then for every η , G_η has positive μ invariant. If we let G_1 be the branch of G with respect to a nontrivial character of Δ_0 , and let G_0 be the branch of G with respect to the trivial character of Δ_0 , then we may divide G_0 and G_1 by p and still

get the power series with integral coefficients. In other words, if we let $G'_i = p^{-1}G_i$, then G'_i lies in $\mathcal{I}[[T]]$ for $i = 0, 1$. However, this is not enough to conclude $G'_0 \equiv G'_1$ modulo p . Indeed, we always have $G'_1 = 0$, while $G'_0 = H(T)$ is not 0 modulo p unless the μ -invariant of $H(T)$ is positive. If we were able to divide G by p inside $\mathcal{I}[\Delta_0][[T]]$ before taking branches, then we would have obtained the congruence between G'_0 and G'_1 . This explains why we need Hypothesis 4.

The next corollary shows that if there is a central zero for $L(f, \rho, s)$ and $L(f, \sigma, s)$, a stronger congruence holds.

COROLLARY 9.2. *Assume that Hypotheses 1, 2, 3 and 4 hold. Further assume that $L(f, \rho, k/2) = 0$ and $L(f, \sigma, k/2) = 0$. Let val_p be the valuation on \mathbb{Z} normalized by $\text{val}_p(p) = 1$, and we put $r = \text{val}_p((1 + p)^{\frac{k}{2}-n} - 1)$. Note that $r \geq 1$ for any integer n . For $n = 1, 2, \dots, k - 1$, we have*

$$\mathcal{L}_p(f, \rho, n) \equiv \mathcal{L}_p(f, \sigma, n) \pmod{p^{1+r}}.$$

Proof. Let θ denote ρ or σ . If we put $F(T) = (1 + p)^{k/2}(T + 1) - 1$, then the assumption implies that $F(T)$ divides $H(\theta, T)$ in $\mathcal{I}[[T]]$. Put $H'(\theta, T) = H(\theta, T)/F(T)$. Now Theorem 9.1 implies that we have $H'(\rho, T) \equiv H'(\sigma, T)$ modulo p . Thus we have $H'(\rho, (1 + p)^{-n} - 1) \equiv H'(\sigma, (1 + p)^{-n} - 1)$ modulo p for $n = 1, 2, \dots, k - 1$. By definition of r we have $F((1 + p)^{-n} - 1) \equiv 0$ modulo p^r , whence the desired congruence follows.

We remark that we could have worked with the field \mathcal{F}_r , the r -th layer of the cyclotomic \mathbb{Z}_p -extension of \mathcal{F} , obtaining an analogue of Theorem 9.1 for \mathcal{F}_r under the hypothesis that the Katz-Hida-Tilouine measure μ_r attached to $\mathcal{F}_r/\mathcal{F}_r^+$ is divisible by p^r , or equivalently that μ_r takes values in $p^r\mathcal{I}$. However, even if we assume that μ_r takes values in $p^r\mathcal{I}$, the congruences predicted by Kato cannot be verified in the way Theorem 9.1 is proved if $r \geq 2$.

10. Numerical example

Let $\mathcal{K} = \mathbb{Q}(\sqrt{-11})$. The elliptic curve E defined by the following equation has complex multiplication by $\mathcal{O}_{\mathcal{K}}$:-

$$E: y^2 + y = x^3 - x^2 - 7x + 10.$$

Furthermore, E has good ordinary reduction at $p = 3$, and $w_{\mathcal{K}} = 1$. Therefore, we may consider f corresponding to ψ_E^3 , which has q -expansion

$$f(q) = q + 8q^3 - 8q^4 + 18q^5 + 37q^9 - 64q^{12} + 144q^{15} + 64q^{16} + \dots$$

where $q = \exp(2\pi iz)$. In [4], the special values of $L(f, \rho, n)$ and $L(f, \sigma, n)$ are computed for $E, k = 4, p = 3$, and cube-free positive integers m up to 20. In this case, the classical periods have well known explicit formulae, going back to Chowla-Selberg, which can be found in [8]. Put

$$\Theta = \Gamma\left(\frac{1}{11}\right) \Gamma\left(\frac{3}{11}\right) \Gamma\left(\frac{4}{11}\right) \Gamma\left(\frac{5}{11}\right) \Gamma\left(\frac{9}{11}\right)$$

and let

$$\Omega_+(f) = \frac{\sqrt{11}\Theta^3}{(2\pi)^9} \text{ and } \Omega_-(f) = \frac{\Theta^3}{(2\pi)^9}$$

Then the canonical period defined in (2.2) can be written as

$$\Omega_{\infty}^{\text{can}}(f, \theta, 1) = \epsilon_p(\theta)^{-1} (2\pi i)^2 \Omega_+(f) \Omega_-(f)$$

where θ denotes ρ or σ . The author is grateful to the authors [4] for sharing the following table of normalized *L*-values. In the following table, when θ is ρ or σ we put

$$\mathcal{P}_3(f, \theta, n) = \prod_{r|3m} P_r(f, \theta, r^{-n}).$$

Moreover, $N(f, \rho)$ denotes the conductor of $L(f, \rho, s)$. As pointed out in [4], $L(f, \sigma, 2) = 0$ and $L(f, \rho, 2) = 0$ for all $m > 1$ by root number considerations. Also, we have $L_3(f, \sigma, 1) = 2^3 \cdot 3$.

<i>m</i>	$L_3(f, \rho, 1)$	$\mathcal{P}_3(f, \rho, 1)$	$\mathcal{P}_3(f, \sigma, 1)$	$N(f, \rho)$
2	$2^4 \cdot 3^2 \cdot 17 \cdot 37$	1	$2^2 \cdot 3$	$2^4 \cdot 3^6 \cdot 11^4$
3	$2^4 \cdot 3 \cdot 5 \cdot 4373$	1	$\frac{2^2}{3}$	$3^{10} \cdot 11^4$
5	$2^4 \cdot 3^4 \cdot 2069$	1	$\frac{2^8 \cdot 3}{5^2}$	$3^6 \cdot 5^4 \cdot 11^4$
6	$2^2 \cdot 3^3 \cdot 83 \cdot 2297$	1	$2^2 \cdot 3$	$2^4 \cdot 3^{10} \cdot 11^4$
7	$\frac{2^4 \cdot 3 \cdot 349 \cdot 863}{7}$	1	$\frac{2^8}{3}$	$3^6 \cdot 7^4 \cdot 11^4$
10	$2^2 \cdot 3^3 \cdot 13 \cdot 211$	$\frac{2^2}{3}$	$\frac{2^8 \cdot 3^3}{5^2}$	$2^4 \cdot 3^2 \cdot 5^4 \cdot 11^4$
11	$2^7 \cdot 3 \cdot 11$	1	$\frac{2^2}{3}$	$3^6 \cdot 11^4$
12	$2^4 \cdot 3^2 \cdot 13 \cdot 31 \cdot 367$	1	$2^2 \cdot 3$	$2^4 \cdot 3^{10} \cdot 11^4$
13	$\frac{2^4 \cdot 3 \cdot 29 \cdot 230281}{13}$	1	$\frac{2^4 \cdot 7^2}{3}$	$3^6 \cdot 11^4 \cdot 13^4$
14	$\frac{2^6 \cdot 3^2 \cdot 439 \cdot 1129}{7}$	1	$2^8 \cdot 3$	$2^4 \cdot 3^6 \cdot 7^4 \cdot 11^4$
15	$\frac{2^4 \cdot 3^2 \cdot 23234851}{5}$	$\frac{2^8 \cdot 3}{5^2}$	1	$3^{10} \cdot 5^4 \cdot 11^4$
17	$\frac{2^4 \cdot 3^2 \cdot 29 \cdot 8219}{17}$	$\frac{2^2}{3}$	$2^4 \cdot 3^3$	$3^2 \cdot 11^4 \cdot 17^4$
19	$\frac{2^4 \cdot 3 \cdot 11 \cdot 29 \cdot 31}{19}$	$\frac{2^2}{3}$	$\frac{2^6 \cdot 5^2}{3}$	$3^2 \cdot 11^4 \cdot 19^4$
20	$\frac{2^6 \cdot 3^3 \cdot 156241}{5}$	1	$\frac{2^8 \cdot 3^3}{5^2}$	$2^4 \cdot 3^6 \cdot 5^4 \cdot 11^4$

<i>m</i>	$\mathcal{L}_3(f, \sigma, 1)$	$\mathcal{L}_3(f, \rho, 1)$
2	$2 \cdot 3^3 + O(3^4)$	$1 \cdot 3^2 + O(3^3)$
3	$2 \cdot 3^1 + O(3^2)$	$2 \cdot 3^1 + O(3^2)$
5	$2 \cdot 3^3 + O(3^4)$	$1 \cdot 3^4 + O(3^5)$
6	$2 \cdot 3^3 + O(3^4)$	$2 \cdot 3^3 + O(3^4)$
7	$2 \cdot 3^1 + O(3^2)$	$2 \cdot 3^1 + O(3^2)$
10	$2 \cdot 3^5 + O(3^6)$	$1 \cdot 3^3 + O(3^4)$
11	$2 \cdot 3^1 + O(3^2)$	$2 \cdot 3^1 + O(3^2)$
12	$2 \cdot 3^3 + O(3^4)$	$2 \cdot 3^2 + O(3^3)$
13	$2 \cdot 3^1 + O(3^2)$	$2 \cdot 3^1 + O(3^2)$
14	$2 \cdot 3^3 + O(3^4)$	$1 \cdot 3^2 + O(3^3)$
15	$2 \cdot 3^3 + O(3^4)$	$1 \cdot 3^2 + O(3^3)$
17	$2 \cdot 3^5 + O(3^6)$	$2 \cdot 3^2 + O(3^3)$
19	$2 \cdot 3^1 + O(3^2)$	$2 \cdot 3^1 + O(3^2)$
20	$2 \cdot 3^5 + O(3^6)$	$1 \cdot 3^3 + O(3^4)$

Note that the authors of [4] uses a different choice of periods in Table IV of [4]. The ratio between the canonical period for this case turns out to be $3/22$, as explained in [4, Example 5.3]. Since both $L(f, \sigma, s)$ and $L(f, \rho, s)$ have central zeros, it is expected that $\mathcal{L}_p(f, \rho, n) \equiv \mathcal{L}_p(f, \sigma, n)$ modulo 3^2 , and the table shows that it is indeed the case. Since 2, 7, 11, 13, 17, 19 are inert or ramified in \mathcal{K} , Corollary 9.2 applies to $m = 2, 7, 11, 13, 14, 17, 19$, providing a theoretical proof for the congruence. Since 3 and 5 split in \mathcal{K} , our method does not apply to $m = 3, 5, 6, 10, 12, 15, 20$.

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