# PATH INTEGRALS AND $p$-ADIC $L$-FUNCTIONS 

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#### Abstract

We prove an arithmetic path integral formula for the inverse $p$-adic absolute values of Kubota-Leopoldt $p$-adic $L$-functions at roots of unity.


## 1. Primes, knots, and quantum fields

1.1. Arithmetic topology. Barry Mazur [17, 18 pointed out long ago that the cohomological properties of $\operatorname{Spec}\left(\mathcal{O}_{F}\right)$, the spectrum of the ring of integers of an algebraic number field, are like those of a 3 -manifold. This went with the observation that the inclusion

$$
\operatorname{Spec}(k(P)) \hookrightarrow \operatorname{Spec}\left(\mathcal{O}_{F}\right)
$$

of the spectrum of the residue field $k(P)$ of a prime $P$ of $\mathcal{O}_{F}$ compares well to the inclusion of a knot $\kappa$ into a 3 -manifold [22]. When we remove a finite collection $S$ of primes and consider $\operatorname{Spec}\left(O_{F}\right) \backslash S$, the properties then are like a 3-manifold with boundary obtained by removing tubular neighbourhoods of the knots. Mazur went on to consider the cases of

$$
\operatorname{Spec}(\mathbb{Z}), \operatorname{Spec}(\mathbb{Z}[1 / p]), \text { and } \operatorname{Spec}\left(\mathbb{Z}\left[\mu_{p}\right]\left[\frac{1}{\zeta_{p}-1}\right]\right)
$$

There, one arrives at an analogy between the covering

$$
\operatorname{Spec}\left(\mathbb{Z}\left[\mu_{p^{\infty}}\right][1 / p]\right) \rightarrow \operatorname{Spec}\left(\mathbb{Z}\left[\mu_{p}\right]\left[\frac{1}{\zeta_{p}-1}\right]\right)
$$

with Galois group

$$
\Gamma:=\operatorname{Gal}\left(\mathbb{Q}\left(\mu_{p^{\infty}}\right) / \mathbb{Q}\left(\mu_{p}\right)\right) \simeq \mathbb{Z}_{p}
$$

[^0]and the maximal abelian covering
$$
D_{\kappa} \rightarrow S^{3} \backslash \kappa,
$$
which has group of deck transformations isomorphic to $\mathbb{Z}$. In Iwasawa theory, the main object of study is
$$
V=\operatorname{Gal}\left(M / \mathbb{Q}\left(\mu_{p^{\infty}}\right)\right),
$$
the Galois group of the maximal abelian unramified $p$-extension $M$ of $\mathbb{Q}\left(\mu_{p} \infty\right)$, acted on by the Iwasawa algebra
$$
\mathbb{Z}_{p}[[\Gamma]] \simeq \mathbb{Z}_{p}[[T]] .
$$

The isomorphism here comes from $\gamma-1 \mapsto T$ for a fixed topological generator $\gamma$ of $\Gamma$. There is also an action of $\operatorname{Gal}\left(\mathbb{Q}\left(\mu_{p}\right) / \mathbb{Q}\right)$ (which can be realised as a subgroup of $\left.\operatorname{Gal}\left(\mathbb{Q}\left(\mu_{p^{\infty}}\right) / \mathbb{Q}\right)\right)$, according to which $V$ splits into isotypic components $V_{k}$ via the Teichmüller character $\omega: \operatorname{Gal}\left(\mathbb{Q}\left(\mu_{p}\right) / \mathbb{Q}\right) \rightarrow \mathbb{Z}_{p}^{\times}$and its powers $\omega^{k}$. The main conjecture of Iwasawa theory [20] as proved by Mazur and Wiles relates the determinants of these isotypic components to various branches of the $p$-adic zeta function. (For general background, we refer the reader to Washington's excellent book [27].) Namely, for an odd prime $p$ and for $j=1,3, \ldots, p-2$ odd, the Kubota-Leopoldt $p$-adic $L$-function $L_{p}\left(\omega^{j}, s\right)$ is the continuous function on $\mathbb{Z}_{p}$ such that

$$
L_{p}\left(\omega^{j}, \ell\right)=\left(1-p^{-\ell}\right) \zeta(\ell)
$$

for negative integers $\ell \equiv j \bmod p-1$. There is a unique power series $z_{j}(T) \in \mathbb{Z}_{p}[[T]]$ such that

$$
z_{j}\left((1+p)^{s}-1\right)=L_{p}\left(\omega^{j}, s\right)
$$

enabling us to identify $z_{j}(T)$ with the $p$-adic $L$-function itself. The main conjecture says that $z_{1-k}(T)$ is the determinant of the $\mathbb{Z}_{p}[[T]]$-module $V_{k}$ for $k \neq 1$ odd. Mazur's main observation in [17] was that the Alexander polynomial of a knot also has a precise definition as a determinant of the module

$$
H_{1}\left(D_{\kappa}, \mathbb{Z}\right)
$$

for

$$
\mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}\left[t, t^{-1}\right]
$$

strengthening the circle of analogies that has now come to be known as arithmetic topology.
1.2. Quantum field theory and knots invariants. Meanwhile, in the late 1980s, Edward Witten [28] gave a remarkable construction of the Jones polynomial of a knot using the methods of quantum field theory, which were then made rigorous by Reshetikhin and Turaev [24]. Here, we have a space $\mathcal{A}$ of $S U(2)$ connections on $S^{3}$ acted upon by a group $\mathcal{G}$ of gauge transformations. The knot $\kappa$ defines a Wilson loop function

$$
W_{\kappa}: \mathcal{A} \rightarrow \mathbb{C}
$$

that sends a connection $A$ to

$$
\operatorname{Tr}\left(\rho\left(\operatorname{Hol}_{\kappa}(A)\right)\right)
$$

the trace of the holonomy of the connection around $\kappa$ evaluated in the standard representation $\rho$ of $S U(2)$. (The importance of such a function should not be surprising at all to number-theorists.) There is also a 'global' Chern-Simons function given by

$$
C S(A)=\frac{1}{8 \pi^{2}} \int \operatorname{Tr}\left(A \wedge d A+\frac{2}{3} A \wedge A \wedge A\right)
$$

which is only gauge-invariant up to integers. Witten's result is then

$$
\int_{\mathcal{A} / \mathcal{G}} W_{\kappa}(A) \exp (2 \pi i n C S(A)) d A=J_{\kappa}\left(\exp \left(\frac{2 \pi i}{n+2}\right)\right)
$$

equating a path integral with the value of the Jones polynomial $J_{\kappa}$ of $\kappa$ at a root of unity. (The reader should beware that many different normalisations exist in the literature.) A somewhat complicated analogue for the Alexander polynomial can be found in [2, 7, 8, 8, 21, 26].
1.3. Arithmetic path integrals. We would like to prove a simple arithmetic analogue of Witten's formula for $p$-adic $L$-functions. Following the framework of arithmetic topological quantum field theory set up in [15, 16, (5, (6, 13, 23], our goal is to represent the $p$-adic $L$-function as an arithmetic path integral, thereby incorporating the perspective of topological quantum field theory into arithmetic topology in a rather concrete fashion and strengthening the analogy envisioned by Mazur. It should be admitted right away that we do not achieve this goal. However, we do find a result about its $p$-adic valuation that appears to be interesting. To describe this, we go on to define the relevant space of 'arithmetic fields'.

Let $q=p^{n}$ where $p$ is an odd prime and $n$ is a positive integer. We set $K=\mathbb{Q}\left(\mu_{q}\right)$ and let

$$
X=\operatorname{Spec}\left(\mathbb{Z}\left[\zeta_{q}\right]\right) \backslash\left(\zeta_{q}-1\right),
$$

where $\zeta_{q}$ is a primitive $q$-th root of unity. We fix an integer $m \geq 1$ and define the space of fields as

$$
\mathcal{F}^{m}:=H^{1}\left(X, \mu_{p^{m}}\right) \times H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)
$$

where $H_{c}^{1}$ denotes compactly supported étale cohomology [19, Chapter 2]. This is an abelian moduli space of principals bundles together with its dual, a setting that allows the definition of topological actions in physics in arbitrary dimension and even on nonorientable manifolds [8]. The BF-action is the map

$$
B F: \mathcal{F}^{m} \rightarrow \frac{1}{p^{m}} \mathbb{Z} / \mathbb{Z}
$$

that takes $(a, b) \in \mathcal{F}^{m}$ to

$$
\operatorname{inv}(d a \cup b)
$$

where

$$
d: H^{1}\left(X, \mu_{p^{m}}\right) \rightarrow H^{2}\left(X, \mu_{p^{m}}\right)
$$

is the Bockstein map coming from the exact sequence

$$
1 \rightarrow \mu_{p^{m}} \rightarrow \mu_{p^{2 m}} \rightarrow \mu_{p^{m}} \rightarrow 1
$$

and

$$
\operatorname{inv}: H_{c}^{3}\left(X, \mu_{p^{m}}\right) \xrightarrow{\sim} \frac{1}{p^{m}} \mathbb{Z} / \mathbb{Z}
$$

is the invariant map [18].
There is a natural action of $G=\operatorname{Gal}(K / \mathbb{Q})$ on the space of fields $\mathcal{F}^{m}$, and we let $G^{\prime}\left(\simeq \operatorname{Gal}\left(\mathbb{Q}\left(\mu_{p}\right) / \mathbb{Q}\right)\right) \subset G$ be the unique subgroup of $G$ of order $p-1$. Since $p-1$ is not divisible by $p, G^{\prime}$ acts semi-simply on $\mathcal{F}^{m}$. Let us define

$$
\mathcal{F}_{k}^{m}:=H^{1}\left(X, \mu_{p^{m}}\right)_{k} \times H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)_{-k}
$$

i.e. on the first factor we take the $\omega^{k}$-eigenspace, and on the second factor we take the $\omega^{-k}$-eigenspace.
Theorem 1.1. Let $k$ be odd and different from 1. Then we have

$$
\left|\prod_{j=0}^{p^{n}-1} z_{1-k}\left(\exp \left(2 \pi i j / p^{n}\right)-1\right)^{-1}\right|_{p}=\lim _{m \rightarrow \infty} \sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b))
$$

Remark 1.2. Note that the variable $T$ here corresponds to $\gamma-1$, where $\gamma$ is a topological generator of $\Gamma$. Thus, the value of $T$ on $\exp \left(2 \pi i j / p^{n}\right)-1$ corresponds to the value $\exp \left(2 \pi i j / p^{n}\right)$ assigned to $\gamma$. In the Alexander polynomial, the variable $t$ indeed corresponds to a generator of $H_{1}\left(S^{3} \backslash K, \mathbb{Z}\right)$. Thus, in the analogy of arithmetic topology, the values occurring on the left of our formula correspond exactly to the values of the standard variable in topology at roots of unity. We could use the Weierstrass preparation theorem and the vanishing of the $\mu$-invariant [10] to replace $z_{j}$ by a distinguished polynomial $P_{j}$, such that

$$
z_{j}(T)=P_{j}(T) u_{j}(T)
$$

for a unit $u_{j}(T)$. With this we can restore the variable $t=T+1$ and put $Q_{j}(t)=$ $P_{j}(t-1)$. The left-hand side of our formula can then be written

$$
\left|\prod_{j=0}^{p^{n}-1} Q_{1-k}\left(\exp \left(2 \pi i j / p^{n}\right)\right)^{-1}\right|_{p}
$$

One might argue that the $Q_{j}$ are the true analogues of the Alexander polynomial.
Remark 1.3. Of course since we are taking the $p$-adic absolute value, the formula is the same if we change $z_{k}$ by a unit. Hence, it really is about characteristic power series rather than the precise choice of $p$-adic $L$-functions. Nevertheless, since the $p$-adic $L$-functions are the objects of central interest in number theory, we have stated the theorem in these terms. Obviously, for this we need the main conjecture of Iwasawa theory which we will take for granted in the rest of this paper.

Remark 1.4. In physics, it's common to be vague about the domain of the path integral. That is, one imagines a sequence of inclusions

$$
\mathcal{C} \subset \mathcal{A}_{1} \subset \mathcal{A}_{2} \subset \cdots
$$

containing the space $\mathcal{C}$ of classical fields (solutions to the equation of motion) and integrals over $\mathcal{A}_{n}$ giving successively more information. On the other hand, after some point, further enlargement shouldn't matter. That is, the inclusion $\mathcal{C} \subset \mathcal{A}_{n}$ into a sufficiently flabby space should play a role similar to an acyclic resolution of a complex where any two resolutions are suitably homotopic. In gauge theory, for example, the space of $C^{\infty}$ connections is thought to be an adequate domain. Even there, one could
include discontinuous or distributional connections in the flavour of Feynman's original heuristic arguments with jagged paths. The limit we are taking can be interpreted as

$$
\int_{\mathcal{F}_{k}} \exp (2 \pi i B F(a, b)) d a d b,
$$

an integral over the domain

$$
\begin{aligned}
& \mathcal{F}_{k}={\underset{\check{m}}{m}}_{\lim ^{1}\left(X, \mu_{p^{m}}\right)_{k} \times \underset{m}{\lim } H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)_{-k}} \\
&=H^{1}\left(X, \mathbb{Z}_{p}(1)\right)_{k} \times H_{c}^{1}\left(X, \mathbb{Q}_{p} / \mathbb{Z}_{p}\right)_{-k} .
\end{aligned}
$$

This has very much the flavour of a space of distributional fields.
Remark 1.5. The obvious challenge is to remove the absolute value from the $p$-adic $L$-value, incorporating the unit information. In the analogy with physics, a unit is like a 'phase', leading one to believe that it can be recovered from a refinement of the given path integral.

## 2. Path integrals for cyclotomic integers

Let $\mathrm{Cl}_{K}$ be the ideal class group of $K$ and $\mathcal{O}_{X}^{\times}$the group of units in $\mathbb{Z}\left[\mu_{q}\right]\left[1 /\left(\zeta_{q}-1\right)\right]$. A repetition of the computations found in [4] shows that the arithmetic path integral

$$
\sum_{(a, b) \in \mathcal{F}^{m}} \exp (2 \pi i B F(a, b))
$$

equals

$$
\left|p^{m} \cdot \mathrm{Cl}_{K}\left[p^{2 m}\right]\right| \cdot\left|\mathcal{O}_{X}^{\times} /\left(\mathcal{O}_{X}^{\times}\right)^{p^{m}}\right| \cdot\left|\mathrm{Cl}_{K} / p^{m}\right| .
$$

Our goal will be to prove an equivariant version of this formula.
With the definitions of the previous section, we have that

$$
\mathcal{F}^{m}=\bigoplus_{k=0}^{p-2} \mathcal{F}_{k}^{m}
$$

We will generally denote by a subscript $(\cdot)_{k}$ the $\omega^{k}$-isotypic component of a $\mathbb{Z}_{p}$-module with $G^{\prime}$-action. We now analyze how the action of $G^{\prime}$ on $\mathcal{F}^{m}$ interacts with the BFfunctional. More precisely, we will see that the BF-functional splits:

$$
\oplus_{k=0}^{p-2} B F_{k}: \bigoplus_{k=0}^{p-2} \mathcal{F}_{k}^{m} \rightarrow \frac{1}{p^{m}} \mathbb{Z} / \mathbb{Z}
$$

To see this, start by letting Div $X$ be the free abelian group generated by the closed points in $X$. Then, as is explained in 44, Section 2], we have that

$$
H^{i}\left(X, \mu_{p^{m}}\right)= \begin{cases}\mu_{p^{m}}(K) & \text { for } i=0  \tag{2.1}\\ Z_{1} / B_{1} & \text { for } i=1 \\ \mathrm{Cl}_{K} / p^{m} & \text { for } i=2 \\ 0 & \text { for } i>2\end{cases}
$$

where

$$
Z_{1}=\left\{(a, I) \in K^{*} \oplus \operatorname{Div} X: \operatorname{div}(a)+p^{m} I=0\right\}
$$

and

$$
B_{1}=\left\{\left(a^{p^{m}},-\operatorname{div}(a)\right) \in K^{*} \oplus \operatorname{Div} X: a \in K^{*}\right\}
$$

As is shown in [1, Section 4], the map $d: H^{1}\left(X, \mu_{p^{m}}\right) \rightarrow H^{2}\left(X, \mu_{p^{m}}\right)$ takes

$$
(a, I) \in H^{1}\left(X, \mu_{p^{m}}\right)
$$

to $I \in \mathrm{Cl}_{K} / p^{m}$. By this observation, it is clear that the map $d$ is equivariant. Let us now note further that the Galois action on

$$
H_{c}^{3}\left(X, \mu_{p^{m}}\right) \underset{\text { inv }}{\simeq} \frac{1}{p^{m}} \mathbb{Z} / \mathbb{Z}
$$

is trivial. This clearly implies that the BF-functional splits into direct sums as claimed, since if

$$
a \in H^{1}\left(X, \mu_{p^{m}}\right)_{i} \quad \text { and } \quad b \in H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)_{j}
$$

we see that $d a \cup b$ lands in the $\omega^{i+j}$-eigenspace of $H_{c}^{3}\left(X, \mu_{p^{m}}\right)$, which is non-zero if and only if $i+j=0(\bmod p-1)$.

By the above analysis of the $G^{\prime}$-action, we see that the sum

$$
\sum_{(a, b) \in \mathcal{F}^{m}} \exp (2 \pi i B F(a, b))
$$

splits:

$$
\prod_{k=0}^{p-2} \sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b))
$$

Using the description of the map $d$ above (in particular that it is equivariant), together with the non-degeneracy of the Artin-Verdier pairing, we find that

## Proposition 2.2.

$$
\sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b))=\left|\left(p^{m} \cdot \mathrm{Cl}_{K}\left[p^{2 m}\right]\right)_{k}\right| \cdot\left|\left(\mathcal{O}_{X}^{\times} /\left(\mathcal{O}_{X}^{\times}\right)^{p^{m}}\right)_{k}\right| \cdot\left|\left(\mathrm{Cl}_{K} / p^{m}\right)_{k}\right|
$$

Proof. If $a \notin \operatorname{ker} d$, then

$$
\sum_{b \in H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)_{-k}} \exp (2 \pi i B F(a, b))=\sum_{b \in H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)_{-k}} \exp (2 \pi i \cdot \operatorname{inv}(d a \cup b))=0
$$

because the Artin-Verdier pairing is non-degenerate. Since $\exp (2 \pi i B F(a, b))=1$ for any $a \in \operatorname{ker} d$, we have

$$
\sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b))=\left|\operatorname{ker} d \cap H^{1}\left(X, \mu_{p^{m}}\right)_{k}\right| \cdot\left|H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)_{-k}\right|
$$

Also, we have

$$
(\operatorname{ker} d)_{k}=\operatorname{ker} d \cap H^{1}\left(X, \mu_{p^{m}}\right)_{k}
$$

since the map $d$ is equivariant. Furthermore, $d$ is the composite of two maps: the surjective map $f_{1}: H^{1}\left(X, \mu_{p^{m}}\right) \rightarrow \mathrm{Cl}_{K}\left[p^{m}\right]$, and then the reduction map

$$
f_{2}: \mathrm{Cl}_{K}\left[p^{m}\right] \rightarrow \mathrm{Cl}_{K} / p^{m}
$$

which are both equivariant as well. The kernel of $f_{1}$ is precisely $\mathcal{O}_{X}^{\times} / p^{m}$, while the kernel of $f_{2}$ equals $p^{m} \cdot \mathrm{Cl}_{K}\left[p^{2 m}\right]$. By taking eigenspaces, we have

$$
\left|(\operatorname{ker} d)_{k}\right|=\left|\left(\mathcal{O}_{X}^{\times} / p^{m}\right)_{k}\right| \cdot\left|\left(p^{m} \cdot \mathrm{Cl}_{K}\left[p^{2 m}\right]\right)_{k}\right| .
$$

Finally, $H_{c}^{1}\left(X, \mathbb{Z} / p^{m}\right)_{-k}$ is dual to $H^{2}\left(X, \mu_{p^{m}}\right)_{k}$ and so the result follows.
It is interesting to realize the path integral

$$
\begin{equation*}
\sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b)) \tag{2.3}
\end{equation*}
$$

as a path integral on

$$
Y:=\operatorname{Spec}(\mathbb{Z}) \backslash\{p\} .
$$

To achieve this, we note that the natural map $\pi: X \rightarrow Y$, factors as

$$
X \xrightarrow{\pi^{\prime}} Y^{\prime} \xrightarrow{\pi^{\prime \prime}} Y
$$

where

$$
Y^{\prime}=\operatorname{Spec}\left(\mathbb{Z}\left[\mu_{p}\right]\right) \backslash\left(\zeta_{p}-1\right) .
$$

If we consider the sheaf $\pi_{*}\left(\mathbb{Z} / p^{m} \mathbb{Z}\right)$, it is easy to see that this sheaf corresponds, under the equivalence between sheaves on $Y$ split by $\pi$ and $G$-modules, to the group ring

$$
\mathbb{Z} / p^{m} \mathbb{Z}[G] \cong \mathbb{Z} / p^{m} \mathbb{Z}[x, y] /\left(x^{p^{m-1}}-1, y^{p-1}-1\right) .
$$

This group ring is isomorphic, as a $G$-module, to

$$
\bigoplus_{k=0}^{p-2}\left(\mathbb{Z} / p^{m} \mathbb{Z}[x] /\left(x^{p^{m-1}}-1\right)\right)_{k}
$$

where the action of $G^{\prime}$ on the $k$-th piece in the direct sum is through $\omega^{k}$. This calculation shows that $\pi_{*}\left(\mathbb{Z} / p^{m} \mathbb{Z}\right)$ splits into a direct sum

$$
\bigoplus_{k=0}^{p-2} \mathcal{M}_{k}
$$

of "eigensheaves" with respect to the $G^{\prime}$-action. Since Cartier duality commutes with pushforward [25, Proposition D.1], we see that

$$
\pi_{*}\left(\mu_{p^{m}}\right)=\bigoplus_{k=0}^{p-2} D\left(\mathcal{M}_{k}\right)
$$

where $D$ denotes the Cartier dual. For $k=0,1, \cdots, p-2$, we now claim that

$$
\mathcal{F}_{k}^{m}=H^{1}\left(Y, D\left(\mathcal{M}_{k}\right)\right) \times H_{c}^{1}\left(Y, \mathcal{M}_{k}\right) .
$$

To see this, it is enough to establish that $H^{1}\left(Y, D\left(\mathcal{M}_{k}\right)\right)$ identifies with the $\omega^{k}$-eigenspace of $H^{1}\left(X, \mu_{p^{m}}\right)$ and that $H_{c}^{1}\left(Y, \mathcal{M}_{k}\right)$ naturally identifies with the $\omega^{-k}$-eigenspace of $H_{c}^{1}\left(X, \mathbb{Z} / p^{m} \mathbb{Z}\right)$.

We proceed by first noting that $\pi_{*}\left(\mu_{p^{m}}\right)=\left(\pi_{*}^{\prime \prime} \circ \pi_{*}^{\prime}\right) \mu_{p^{m}}$, and that since $\left|G^{\prime}\right|$ is prime to $p$, taking $G^{\prime}$-fixed points is an exact functor. Then we have the following string of equalities:
$H^{0}\left(G^{\prime}, H^{i}\left(X, \mu_{p^{m}}\right)\right)=H^{0}\left(G^{\prime}, H^{i}\left(Y^{\prime}, \pi_{*}^{\prime} \mu_{p^{m}}\right)\right)=H^{0}\left(G^{\prime}, H^{i}\left(Y^{\prime}, D\left(\mathcal{M}_{0}\right)\right)\right)=H^{i}\left(Y, D\left(\mathcal{M}_{0}\right)\right)$.

The first equality follows from the fact that $\pi^{\prime}$ is finite étale, the second follows from the fact that the restriction of $D\left(\mathcal{M}_{0}\right)$ to $Y^{\prime}$ is isomorphic to $\pi_{*}^{\prime} \mu_{p^{m}}$, and the last follows from the fact that taking $G^{\prime}$-fixed points is an exact functor. This shows that $H^{i}\left(Y, D\left(\mathcal{M}_{0}\right)\right)$ naturally identifies with the $\omega^{0}$-eigenspace of $H^{i}\left(X, \mu_{p^{m}}\right)$, and we proceed by analysing the other eigenspaces. Let us note that

$$
D\left(\mathcal{M}_{k}\right)=D\left(\mathcal{M}_{0}\right) \otimes \mathcal{G}_{k},
$$

where $\mathcal{G}_{k}$ is the sheaf which, under the equivalence between sheaves split by $\pi^{\prime \prime}$ and $G^{\prime}$-modules, corresponds to $\mathbb{Z} / p^{m} \mathbb{Z}$, but where the action of $G^{\prime}$ is through $\omega^{-k}$. We now claim that

$$
H^{i}\left(Y^{\prime}, D\left(\mathcal{M}_{k}\right)\right)=H^{i}\left(Y^{\prime}, D\left(\mathcal{M}_{0}\right)\right)\left(\omega^{-k}\right)
$$

where $H^{i}\left(Y, D\left(\mathcal{M}_{0}\right)\right)\left(\omega^{-k}\right)$ is just $H^{i}\left(Y^{\prime}, D\left(\mathcal{M}_{0}\right)\right)$ as an abelian group, but with the $G^{\prime}$-action twisted by $\omega^{-k}$. Indeed, the pullback of

$$
D\left(\mathcal{M}_{0}\right) \otimes \mathcal{G}_{k}
$$

to $Y^{\prime}$ is isomorphic, as an abelian sheaf, to $D\left(\mathcal{M}_{0}\right)$, and looking at the Cech complex one finds that $H^{i}\left(Y^{\prime}, D\left(\mathcal{M}_{0}\right)\right)\left(\omega^{-k}\right)$ is indeed isomorphic to $H^{i}\left(Y^{\prime}, D\left(\mathcal{M}_{k}\right)\right)$.

Thus,
$H^{0}\left(G^{\prime}, H^{i}\left(X, \mu_{p^{m}}\right)\left(\omega^{-k}\right)\right)=H^{0}\left(G^{\prime}, H^{i}\left(Y^{\prime}, \pi_{*}^{\prime} \mu_{p^{m}}\right)\left(\omega^{-k}\right)\right)=H^{0}\left(G^{\prime}, H^{i}\left(Y^{\prime}, D\left(M_{0}\right)\right)\left(\omega^{-k}\right)\right)$
which equals $H^{0}\left(G^{\prime}, H^{i}\left(Y^{\prime}, D\left(\mathcal{A}_{k}\right)\right)\right)=H^{i}\left(Y, D\left(\mathcal{M}_{k}\right)\right)$. Since

$$
H^{i}\left(X, \mu_{p^{m}}\right)_{k}=H^{0}\left(G^{\prime}, H^{i}\left(X, \mu_{p^{m}}\right)\left(\omega^{-k}\right)\right)
$$

this gives that $H^{i}\left(X, \mu_{p^{m}}\right)_{k}=H^{i}\left(Y, D\left(\mathcal{M}_{k}\right)\right)$, which is the equality we wished to prove. Repeating the same argument for $\mathcal{M}_{k}$ then shows that

$$
\mathcal{F}_{k}^{m}=H^{1}\left(Y, D\left(\mathcal{M}_{k}\right)\right) \times H_{c}^{1}\left(Y, \mathcal{M}_{k}\right),
$$

as claimed.
The realization of the path integral (2.3) as a path integral on $Y$ is now straightforward. Indeed, there is a natural Bockstein map $d: H^{1}\left(Y, D\left(\mathcal{M}_{k}\right)\right) \rightarrow H^{2}\left(Y, D\left(\mathcal{N}_{k}\right)\right)$ and we define

$$
B F: \mathcal{F}_{k}^{m}=H^{1}\left(Y, D\left(\mathcal{M}_{k}\right)\right) \times H_{c}^{1}\left(Y, \mathcal{M}_{k}\right) \rightarrow \frac{1}{p^{m}} \mathbb{Z} / \mathbb{Z}
$$

as $B F(a, b)=\operatorname{inv}(d a \cup b)$. This realizes the path integral as a path integral on $Y$, which is what we wanted to see.

## 3. Calculation for large values of $m$

We analyse the right-hand-side of the formula in Proposition 2.1 for large values of $m$. If $p^{m}>\left|\mathrm{Cl}_{K}\right|$, then the first factor is one and the third factor equals the size of the $\omega^{k}$-isotypic component of the $p$-primary part of $\mathrm{Cl}_{K}$. Below, we look into the factor in the middle. We will assume for simplicity that $p^{m}>\left|\mathrm{Cl}_{K}\left[p^{\infty}\right]\right|$.

Let $U^{\prime} \subset \mathcal{O}_{X}^{\times}$be the subgroup generated by a primitive $q$-th root of unity $\zeta$ and elements of the form $1-\zeta^{a}$ where $a=q, 2,3, \cdots, q-1$. Therefore, we have a finite quotient $A:=\mathcal{O}_{X}^{\times} / U^{\prime}$.

From the exact sequence

$$
0 \rightarrow U^{\prime} \rightarrow \mathcal{O}_{X}^{\times} \rightarrow A \rightarrow 0
$$

and the snake lemma, we get the long exact sequence

$$
\begin{gathered}
0 \rightarrow U^{\prime}\left[p^{m}\right] \rightarrow \mathcal{O}_{X}^{\times}\left[p^{m}\right] \rightarrow A\left[p^{m}\right] \\
\rightarrow U^{\prime} /\left(U^{\prime}\right)^{p^{m}} \rightarrow \mathcal{O}_{X}^{\times} /\left(\mathcal{O}_{X}^{\times}\right)^{p^{m}} \rightarrow A / A^{p^{m}} \rightarrow 0 .
\end{gathered}
$$

Since

$$
U^{\prime}\left[p^{m}\right] \simeq \mathcal{O}_{X}^{\times}\left[p^{m}\right] \simeq \mu_{p^{n}}
$$

for $m \geq n$, this gives an exact sequence

$$
0 \rightarrow A\left[p^{m}\right] \rightarrow U^{\prime} /\left(U^{\prime}\right)^{p^{m}} \rightarrow \mathcal{O}_{X}^{\times} /\left(\mathcal{O}_{X}^{\times}\right)^{p^{m}} \rightarrow A / A^{p^{m}} \rightarrow 0
$$

and the same after taking $\omega^{k}$-isotypic components:

$$
0 \rightarrow A\left[p^{m}\right]_{k} \rightarrow\left(U^{\prime} /\left(U^{\prime}\right)^{p^{m}}\right)_{k} \rightarrow\left(\mathcal{O}_{X}^{\times} /\left(\mathcal{O}_{X}^{\times}\right)^{p^{m}}\right)_{k} \rightarrow\left(A / A^{p^{m}}\right)_{k} \rightarrow 0
$$

Note that $A\left[p^{m}\right]_{k} \simeq A_{k}\left[p^{m}\right]$ and $\left(A / A^{p^{m}}\right)_{k} \simeq A_{k} / A_{k}^{p^{m}}$ as $\left|G^{\prime}\right|$ has order prime to $p$. Since $A$ and hence $A_{k}$ is finite, the kernel and cokernel have the same order, so that

$$
\left|\left(U^{\prime} /\left(U^{\prime}\right)^{p^{m}}\right)_{k}\right|=\left|\left(\mathcal{O}_{X}^{\times} /\left(\mathcal{O}_{X}^{\times}\right)^{p^{m}}\right)_{k}\right| .
$$

Put $U:=U^{\prime} / \mu_{q}$. Since $U$ is torsion-free, another easy snake lemma argument gives an exact sequence

$$
0 \rightarrow \mu_{q} \rightarrow U^{\prime} /\left(U^{\prime}\right)^{p^{m}} \rightarrow U / U^{p^{m}} \rightarrow 0
$$

for $m \geq n$. Thus, for $k \neq 1$, we get an isomorphism

$$
\left(U^{\prime} /\left(U^{\prime}\right)^{p^{m}}\right)_{k} \simeq\left(U / U^{p^{m}}\right)_{k}
$$

Lemma 3.1. We have

$$
\left(U^{\prime} /\left(U^{\prime}\right)^{p^{m}}\right)_{k}= \begin{cases}\{1\} & \text { if } k \text { is odd and } k \neq 1, \\ \mathbb{Z} / p^{m} \mathbb{Z} & \text { if } k \text { is even. }\end{cases}
$$

Proof. The structure of $U$ as a Galois module is known. Let $K^{+}$be the maximal totally real subfield of $K$ and let $G^{+}=\operatorname{Gal}\left(K^{+} / \mathbb{Q}\right)$. In 3, Theorem 3], Bass proved that there is an isomorphism $U \simeq \mathbb{Z}\left[G^{+}\right]$as Galois modules. Thus, the assertion follows since $\mathbb{Z} / p^{m} \mathbb{Z}\left[G^{+}\right]=\bigoplus_{k: \text { even }}\left(\mathbb{Z} / p^{m} \mathbb{Z}\left[G^{+}\right]\right)_{k}$ and each summand is isomorphic to $\mathbb{Z} / p^{m} \mathbb{Z}$.

As a consequence, for even $k$, we get

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \frac{1}{p^{m}} \sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b))=\left|\mathrm{Cl}_{K}\left[p^{\infty}\right]_{k}\right| \tag{3.2}
\end{equation*}
$$

and for odd $k \neq 1$,

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b))=\left|\mathrm{Cl}_{K}\left[p^{\infty}\right]_{k}\right| \tag{3.3}
\end{equation*}
$$

## 4. Connection to $p$-adic $L$-functions: Proof of Theorem 1.1.

We interpret the right-hand side of the formula in terms of special values of the Kubota-Leopoldt $p$-adic $L$-functions. As in the introduction, let

$$
V=\operatorname{Gal}\left(M / \mathbb{Q}\left(\mu_{p} \infty\right)\right),
$$

the Galois group of the maximal abelian unramified $p$-extension $M$ of $\mathbb{Q}\left(\mu_{p} \infty\right)$. We now vary $n$ in $q=p^{n}$, and denote by $\mathrm{Cl}_{K_{n}}\left[p^{\infty}\right]$ the $p$-primary part of the ideal class group of $K_{n}:=\mathbb{Q}\left(\mu_{p^{n+1}}\right)$. We assume in the following that $k \neq 1$ is odd. By the main conjecture and the vanishing of the $\mu$-invariant [10], we have an exact sequence

$$
0 \rightarrow A \rightarrow V_{k} \rightarrow M \rightarrow B \rightarrow 0
$$

where $A$ and $B$ are finite,

$$
M \simeq \prod_{j} \mathbb{Z}_{p}[[T]] /\left(f_{j}(T)\right)
$$

and the $f_{j}(T)$ are power series such that

$$
z_{1-k}(T)=\prod_{j} f_{j}(T)
$$

Consider the action of $\Gamma_{n} \subset \Gamma$ generated by $\gamma^{p^{n}}$, where $\gamma$ is a topological generator of $\Gamma$. We get an equation of $\Gamma_{n}$-Euler characteristics

$$
\frac{\chi\left(\Gamma_{n}, A\right) \chi\left(\Gamma_{n}, M\right)}{\chi\left(\Gamma_{n}, V_{k}\right) \chi\left(\Gamma_{n}, B\right)}=1
$$

where

$$
\chi\left(\Gamma_{n}, \cdot\right)=\prod\left|H_{i}\left(\Gamma_{n}, \cdot\right)\right|^{(-1)^{i}}
$$

Note that

$$
H_{i}\left(\Gamma_{n}, \cdot\right) \simeq H^{i}\left(\Gamma_{n},(\cdot)^{\vee}\right)^{\vee},
$$

where

$$
(\cdot)^{\vee}=\operatorname{Hom}(\cdot, \mathbb{Q} / \mathbb{Z})
$$

is the Pontriagin dual. On the other hand, since $A$ and $B$ are finite, the exact sequence

$$
0 \rightarrow H^{0}\left(\Gamma_{n}, A^{\vee}\right) \rightarrow A \xrightarrow{{p^{p^{n}}}_{\longrightarrow}^{1}} A \rightarrow H^{1}\left(\Gamma_{n}, A^{\vee}\right) \rightarrow 0
$$

and the similar one for $B$ imply that

$$
\chi\left(\Gamma_{n}, A\right)=\chi\left(\Gamma_{n}, B\right)=1
$$

Therefore,

$$
\chi\left(\Gamma_{n}, V_{k}\right)=\chi\left(\Gamma_{n}, M\right)
$$

On the other hand, since all the extensions $K_{n} / \mathbb{Q}\left(\mu_{p}\right)$ are totally ramified over the single prime lying above $p$, by [11, Section 1] we get

$$
\left(\mathrm{Cl}_{K_{n}}\left[p^{\infty}\right]\right)_{k} \simeq V_{k} /\left((T+1)^{p^{n}}-1\right) V_{k}=H_{0}\left(\Gamma_{n}, V_{k}\right)
$$

By [27, Corollary 13.29], we have

$$
V_{k} \simeq \mathbb{Z}_{p}^{a}
$$

for some $a \geq 0$. We have the exact sequence

$$
0 \rightarrow H_{1}\left(\Gamma_{n}, V_{k}\right) \rightarrow V_{k} \xrightarrow{\gamma^{p^{n}}-1} V_{k} \rightarrow H_{0}\left(\Gamma_{n}, V_{k}\right) \rightarrow 0
$$

Since $H_{0}\left(\Gamma_{n}, V_{k}\right)$ is finite, the operator $\gamma^{p^{n}}-1$ does not have the eigenvalue 0 . Hence,

$$
H_{1}\left(\Gamma_{n}, V_{k}\right)=0 .
$$

Similarly, $H_{1}\left(\Gamma_{n}, M\right)=0$. Thus,

$$
\left|V_{k} /\left((T+1)^{p^{n}}-1\right) V_{k}\right|=\left|H_{0}\left(\Gamma_{n}, V_{k}\right)\right|=\left|H_{0}\left(\Gamma_{n}, M\right)\right|=\left|M /\left((T+1)^{p^{n}}-1\right) M\right| .
$$

So finally,

$$
\begin{aligned}
\left|\left(\mathrm{Cl}_{K_{n}}\left[p^{\infty}\right]\right)_{k}\right| & =\left|V_{k} /\left((T+1)^{p^{n}}-1\right) V_{k}\right| \\
& =\left|\mathbb{Z}_{p}[[T]] /\left((T+1)^{p^{n}}-1, z_{1-k}(T)\right)\right| \\
& =\left|\prod_{j} \mathbb{Z}_{p} /\left(z_{1-k}\left(\zeta_{p^{n}}^{j}-1\right)\right)\right| \\
& =\left|\prod_{j} z_{1-k}\left(\zeta_{p^{n}}^{j}-1\right)^{-1}\right|_{p}
\end{aligned}
$$

yielding the desired formula.
Remark 4.1. For even values of $k$, (3.2) implies the vanishing

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \frac{1}{\sum_{(a, b) \in \mathcal{F}_{k}^{m}} \exp (2 \pi i B F(a, b))}=0, \tag{4.2}
\end{equation*}
$$

which is superficially analogous to the vanishing of $L(\chi, s)$ at all negative integers when $\chi$ is an odd Dirichlet character.

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