Mathematical and Numerical Optimization, 3341.454
E. Ryu
Fall 2020

Homework 10

Problem 1: *Contractive* $\subset$ *averaged* $\subset$ *nonexpansive.* Let $R < 1$. Show

$$\mathcal{L}_R \subset \mathcal{N}_{\frac{1+R}{2}} \subset \mathcal{L}_1.$$

Problem 2: Show $\mathbb{A} \in \mathcal{N}_{1/2} \;\Leftrightarrow\; \mathbb{A} \in \mathcal{C}_1 \;\Leftrightarrow\; \mathbb{I} - \mathbb{A} \in \mathcal{N}_{1/2} \;\Leftrightarrow\; 2\mathbb{A} - \mathbb{I} \in \mathcal{L}_1$.

Problem 3: Let $L > 0$. Consider

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 = \mathbb{F}(x),$$

where $\mathbb{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ is monotone and L-Lipschitz. Using $\mathcal{G}(\mathbb{I} - \alpha(\mathcal{M} \cap \mathcal{L}_L))$, explain why is it not possible to establish convergence of the forward step method

$$x^{k+1} = x^k - \alpha \mathbb{F} x^k$$

without further assumptions.

Problem 4: Show that $\mathbb{A} \in \mathcal{N}_\theta$ if and only if $\mathbb{I} - \mathbb{A} \in \mathcal{C}_{1/(2\theta)}$.
Remark. We had proved this result in the proof of Theorem 2, but a proof using the SRG provides geometric intuition.

Problem 5: *Optimal parameter for gradient descent.* Let $0 < \mu < L < \infty$. We previously established that $\mathbb{I} - \alpha\partial\mathcal{F}_{\mu,L} \subseteq \mathcal{L}_R$ with

$$R = \max\{|1 - \alpha\mu|, |1 - \alpha L|\},$$

which provides an exponential rate of convergence for the gradient method

$$x^{k+1} = x^k - \alpha\nabla f(x^k).$$

What is the optimal choice of $\alpha > 0$ that minimizes the contraction factor? Describe $\mathcal{G}(\mathbb{I} - \alpha\partial\mathcal{F}_{\mu,L})$ with the optimal α.

Problem 6: *Nonexpansive and inverse Lipschitz residual makes Krasnosel'skiĭ–Mann contractive.* Show that if $\mathbb{T}$ is nonexpansive and $(\mathbb{I} - \mathbb{T})^{-1}$ is γ-Lipschitz, with $\gamma \geq 1/2$ and $\theta \in (0, 1)$, then

$$(1 - \theta)\mathbb{I} + \theta\mathbb{T} \in \mathcal{L}\left(\sqrt{1 - \frac{\theta(1-\theta)}{\gamma^2}}\right).$$

Problem 7: *Proximal point with inverse Lipschitz operator.* Let $\alpha, \gamma \in (0, \infty)$. Show that if $\mathcal{A} = \mathcal{L}_\gamma^{-1} \cap \mathcal{M}$, then $J_{\alpha\mathcal{A}} \subseteq \mathcal{L}_R$ for

$$R = \frac{\gamma}{\sqrt{\alpha^2 + \gamma^2}}.$$

Also show that the result is tight in the sense that $J_{\alpha\mathcal{A}} \nsubseteq \mathcal{L}_R$ for any smaller value of R.

Problem 8: *SRG of DRS.* Let $\mathcal{A}$ be an SRG-full operator class such that $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{L}_R)$ and R is tight in the sense that there exists a $z \in \mathcal{G}(\mathcal{A})$ such that $|z| = R$. Show that $\mathcal{G}(\mathcal{A}\mathcal{L}_1) = \mathcal{G}(\mathcal{L}_1\mathcal{A}) = \mathcal{G}(\mathcal{L}_R)$. Also show that

$$\mathcal{G}\left(\frac{1}{2}\mathbb{I} + \frac{1}{2}\mathcal{A}\mathcal{L}_1\right) = \mathcal{G}\left(\frac{1}{2}\mathbb{I} + \frac{1}{2}\mathcal{L}_1\mathcal{A}\right) \subseteq \mathcal{G}\left(\mathcal{L}_{\frac{1}{2}+\frac{1}{2}R}\right).$$

Remark. Since the SRG-full classes are defined to contain operators on $\mathbb{R}^n$ for all $n \geq 1$, it is sufficient to consider the case $n = 2$ and appeal to Lemma 4.

Problem 9: Complete the proof of Theorem 20. Specifically, given $\mathcal{G}(\mathcal{F}_{0,\infty}) = \{z \mid \operatorname{Re} z \geq 0\} \cup \{\infty\}$, prove the characterizations of $\mathcal{G}(\partial\mathcal{F}_{\mu,\infty})$, $\mathcal{G}(\partial\mathcal{F}_{0,L})$, and $\mathcal{G}(\partial\mathcal{F}_{\mu,L})$ asserted in Theorem 20.

Hint. Use the facts $\partial\mathcal{F}_{\mu,\infty} = \mu\mathbb{I} + \partial\mathcal{F}_{0,\infty}$, $\partial\mathcal{F}_{0,L} = \left(\partial\mathcal{F}_{1/L,\infty}\right)^{-1}$, and $\partial\mathcal{F}_{\mu,L} = \mu\mathbb{I} + \partial\mathcal{F}_{0,L-\mu}$.