Weingarten calculus on computers and its application to random quantum Gaussian states.

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Contents

1 Weingarten Calculus on computers

"RTNI - A symbolic integrator for Haar-random tensor networks",
arXiv:1902.08539 [quant-ph] with Nechita and Koenig.

1.1 Computer packages for average over Haar unitary matrices

Preceding packages:

- `IntU`, a Mathematica package, [Puchała and Miszczak (2017)].
- `IntHaar`, a Maple package, [Ginory and Kim (2016)]. (includes Haar orthogonal and symplectic cases.)

Our package:

`RTNI` (Random Tensor Network Integrator),
Mathematica and Python packages.

Differences:

- Others calculate averages of monomials in the entries of a random unitary matrices.
- Our package calculates averages symbolically, which allows tensor structures easily.

1.2 Polynomial version of Weingarten calculus

$\mathcal{U}(n)$: group of $n \times n$ unitary matrices, and
dU: normalized Haar measure.

Fix $p \in \mathbb{N}$ and let $i = (i_1, \ldots, i_p)$, $i' = (i'_1, \ldots, i'_p)$, $j = (j_1, \ldots, j_p)$, $j' = (j'_1, \ldots, j'_p)$ be p-tuples of positive integers from $\{1, 2, \ldots, n\}$. [Collins and Sniady (2006)]

$$\int_{\mathcal{U}(n)} U_{i_1 j_1} \cdots U_{i_p j_p} \bar{U}_{i'_1 j'_1} \cdots \bar{U}_{i'_p j'_p} \mathrm{d}U$$
$$= \sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \, \mathrm{Wg}_n(\alpha^{-1}\beta) \ .$$

Here, $\mathcal{S}_p$ is the permutation group and $\mathrm{Wg}_n(\cdot)$ is called the *unitary Weingarten function*. Also, note that

$$\int_{\mathcal{U}(n)} U_{i_1 j_1} \cdots U_{i_p j_p} \bar{U}_{i'_1 j'_1} \cdots \bar{U}_{i'_{p'} j'_{p'}} \mathrm{d}U = 0 \qquad \text{for} \quad p \neq p' \ .$$

For integer $n \geq p$, the Weingarten functions can be written as

$$\mathrm{Wg}_n(\sigma) = \frac{1}{(p!)^2} \sum_{\lambda \vdash p} \frac{(\chi^\lambda(e))^2}{s_{\lambda,n}(1)} \chi^\lambda(\sigma) \ .$$

- $\lambda \vdash p$ means that λ is a partition of the integer p.
- χ^λ is the character of the irreducible representation of the symmetric group $\mathcal{S}_p$ specified by λ.
- $s_{\lambda,n}(1)$ is the Schur polynomial evaluated at the identity:

$$s_{\lambda,n}(1) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{i - j} \ .$$

Use Murnaghan–Nakayama rule to generate the character tables of $\mathcal{S}_p$.

- Non-recursive method: Young tableau.
- Recursive method: Young diagrams.

Examples.

$$\mathrm{Wg}_n((1)) = \frac{1}{n}, \quad \mathrm{Wg}_n((1,1)) = \frac{1}{n^2-1}, \quad \mathrm{Wg}_n((2)) = \frac{-1}{n(n^2-1)}.$$

$$\int_{\mathcal{U}(n)} U_{ij}\bar{U}_{i'j'}\mathrm{d}U = \sum_{\alpha,\beta\in\mathcal{S}_1} \delta_{i_1 i'_{\alpha(1)}}\delta_{j_1 j'_{\beta(1)}} \mathrm{Wg}_n(\alpha^{-1}\beta) = \frac{1}{n}\delta_{ii'}\delta_{jj'} \ .$$

$$\begin{aligned}\mathbb{E}[(UAU^*)_{k\ell}] &= \sum_{r,s=1}^{n} \mathbb{E}[U_{kr}A_{rs}U^*_{s\ell}] = \sum_{r,s=1}^{n} \mathbb{E}[U_{kr}A_{rs}\bar{U}_{\ell s}] \\ &= \sum_{r,s=1}^{n} \frac{1}{n}\delta_{k\ell}\,\delta_{rs}\,A_{rs} \ .\end{aligned}$$

$$\mathbb{E}[(UA\bar{U})_{k\ell}] = \sum_{r,s=1}^{n} \mathbb{E}[U_{kr}A_{rs}\bar{U}_{s\ell}] = \sum_{r,s=1}^{n} \frac{1}{n}\delta_{ks}\,\delta_{r\ell}\,A_{rs} \ .$$

1.3 Graphical version

Examples - continued.

$$\mathbb{E}[(UAU^*)_{k\ell}] = \sum_{r,s=1}^{n} \frac{1}{n}\delta_{k\ell}\,\delta_{rs}\,A_{rs} = \frac{1}{n}\delta_{k\ell}\sum_{r=1}^{n} A_{rr}\ .$$

This means that

$$\mathbb{E}[UAU^*] = \frac{\mathrm{Tr}[A]}{n} I_n\ .$$

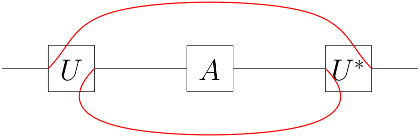
$$\mathbb{E}[(UA\bar{U})_{k\ell}] = \sum_{r,s=1}^{n} \frac{1}{n}\delta_{ks}\,\delta_{r\ell}\,A_{rs} = \sum_{r,s=1}^{n} \frac{1}{n} A_{lk}\ .$$

This means that

$$\mathbb{E}[UAU^*] = \frac{1}{n} A^T.$$

Graphical calculus [Collins and Nechita (2011)]

$$\mathbb{E}[UAU^*] = \frac{\mathrm{Tr}[A]}{n} I_n$$



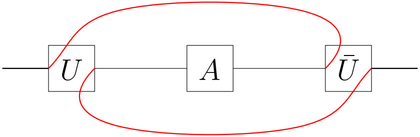
Original contractions:

[L-end, $(U,$ out$)$], [$(U,$ in$)$, $(A,$ out$)$], [$(A,$ in$)$, $(U^*,$ out$)$], [$(U^*,$ in$)$, R-end]

After:

[L-end, R-end], [[A, out], [A, in]]

$$\mathbb{E}[UAU^*] = \frac{1}{n} A^T$$



Original contractions:

[L-end, $(U,$ out$)$], [$(U,$ in$)$, $(A,$ out$)$], [$(A,$ in$)$, $(U^*,$ in$)$], [$(U^*,$ out$)$, R-end]

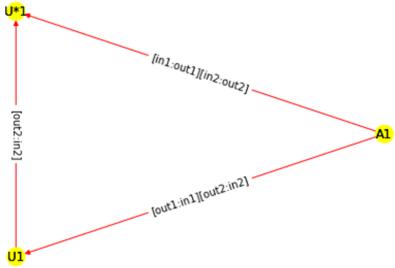
After:

[L-end, [A, in]], [[A, out], R-end]

1.4 "Computer version"

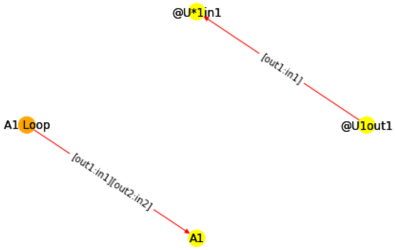
$[\mathrm{id} \otimes \mathrm{Tr}](UAU^*)$:

```
In [1]: from IHU_source import *
In [2]: e1 = [["A", 1, "out", 1], ["U", 1, "in", 1]]
In [3]: e2 = [["A", 1, "out", 2], ["U", 1, "in", 2]]
In [4]: e3 = [["U*", 1, "out", 1], ["A", 1, "in", 1]]
In [5]: e4 = [["U*", 1, "out", 2], ["A", 1, "in", 2]]
In [6]: e5 = [["U", 1, "out", 2], ["U*", 1, "in", 2]]
In [7]: g = [e1, e2, e3, e4, e5]
In [8]: gw = [g,1]
In [9]: visualizeTN(gw)
```



$\mathbb{E}[[\mathrm{id} \otimes \mathrm{Tr}](UAU^*)]$:

```
In [10]: k,n = symbols('k n')
In [11]: rm = ["U",[n,k],[n,k],n*k]
In [12]: Eg = integrateHaarUnitary(gw,rm)
In [13]: print(Eg)
In [14]: visualizeTN(Eg)
Out[1]: [[[[['@U*', 1, 'in', 1], ['@U', 1, 'out', 1]],
[['A', 1, 'out', 1], ['A', 1, 'in', 1]],
[['A', 1, 'out', 2], ['A', 1, 'in', 2]]], 1/n]]
```



This figure is followed by the new weight $1/n$ in the program.

1.5 Demo (had been done.)

To get RTNI, go to
`https://github.com/MotohisaFukuda/RTNI`

To get a user-friendly but limited version, called RTNI_light, go to
`https://github.com/MotohisaFukuda/RTNI_light`

To try a web version of RTNI_light, go to
`https://motohisafukuda.pythonanywhere.com`

2 Random Gaussian states

"Typical entanglement for Gaussian states", arXiv:1903.04126 [quant-ph] with Koenig.

2.1 Motivation and past research

Typical entanglement entropy of random bipartite quantum states.

- Lubkin('78); Lloyd & Pagels('88); Page('93); Foong & Kanno('94): average of entanglement entropy.
- Hayden, Leung and Winter ('04): concentration of entanglement entropy.

In the setting of random Gaussian states

- Serafini, Dahlsten, Gross & Plenio ('07):
  - micto-canonical measure and canonical measure.
  - inverse squared purity for subsystem of one mode is calculated.
- Us:
  - direct access to symplectic eigenvalues
  - the number of modes of subsystem can grow as n^κ with $\kappa < 1$ with the total mode is n, in the regime $n \to \infty$.
- Others:

2.2 Passive Gaussian unitary operations

The set of Gaussian unitary maps has one-to-one correspondence to the real symplectic group $\mathsf{Sp}(2n) = \{S \in M_{2n\times 2n}(\mathbb{R}) \mid SJS^T = J\}$; a Gaussian unitary evolution of a Gaussian state means, for $S \in \mathsf{Sp}(2n)$

$$M \mapsto SMS^T$$

where M is the covariance matrix of the state.

Replacing the real symplectic group $\mathsf{Sp}(2n)$ by the orthogonal symplectic group $\mathsf{K}(2n) = \mathsf{Sp}(2n) \cap \mathsf{O}(2n)$, which is compact. The map

$$\mathsf{U}(n) \to \mathsf{K}(2n)$$

$$U \mapsto \begin{pmatrix} \mathsf{Re}(U) & \mathsf{Im}(U) \\ -\mathsf{Im}(U) & \mathsf{Re}(U) \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix}\begin{pmatrix} I_n & -iI_n \\ -iI_n & I_n \end{pmatrix}$$

induces the measure on $\mathsf{K}(2n)$.

2.3 Partial trace as Gaussian operation

Suppose we have a Gaussian state of n-mode and want to get the covariance matrix of the reduced k-mode state (with redundant spaces).

For the covariance ($2n \times 2n$) matrix M_n of the n-mode, calculate

$$M_k = \hat{\Pi}_{n,k} M_n \hat{\Pi}_{n,k}$$

with the projection

$$\hat{\Pi}_{n,k} = \begin{pmatrix} \Pi_{n,k} & 0_n \\ 0_n & \Pi_{n,k} \end{pmatrix},$$

where $\Pi_{n,k} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{n-k \text{ times}})$ is a projection matrix of rank k.

2.4 Symplectic eigenvalues

A valid covariance matrix ($M + iJ \geq 0$) has *Williamson normal form*: there is $S \in \mathsf{Sp}(2n) = \{S \in M_{2n\times 2n}(\mathbb{R}) \mid SJS^T = J\}$ such that

$$SMS^T = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n) \ .$$

where λ_j satisfy $\lambda_j \geq 1$ and are called the *symplectic eigenvalues* .

They can be obtained by computing the spectrum of the matrix JM, which consists of complex conjugate pairs as follows:

$$\mathsf{spec}(JM) = \bigcup_{j=1}^{n} \{\pm i\lambda_j\} \ .$$

Note that $J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$.

2.5 Random Gaussian pure states

The covariance matrix M of a pure n-mode Gaussian state can be diagonalized by Euler/Bloch-Messiah decomposition;

$$M = S \begin{pmatrix} Z_n & 0 \\ 0 & Z_n^{-1} \end{pmatrix} S^T \qquad \text{for some} \qquad S \in \mathsf{K}(2n) \ .$$

Here, $Z_n = \mathrm{diag}(z_1, \ldots, z_n)$ with $z_j \geq 1$.

The induced measure on $\mathsf{K}(2n)$ generates random pure Gaussian states.

We call $\{z_j\}_{j=1}^n$ the *squeezing parameters*. Observe that such a state has energy:

$$\mathrm{Tr}\left[Z_n + Z_n^{-1}\right] = \sum_{j=1}^{n} E_j \quad \text{where} \quad E_j = \frac{1}{2}(z_j + 1/z_j) \ .$$

We can understand E_j as the energy in the j-th mode after a suitable Gaussian unitary rotation.

2.6 Symplectic eigenvalues of reduced states

Let M_k be the covariant matrix of k-mode, reduced from the random state of n mode. Then,

$$JM_k = \frac{1}{2}\begin{pmatrix} i\Pi & \Pi \\ -\Pi & -i\Pi \end{pmatrix}\begin{pmatrix} UAU^T & -iUBU^* \\ -i\bar{U}BU^T & -\bar{U}AU^* \end{pmatrix}\begin{pmatrix} \Pi & i\Pi \\ i\Pi & \Pi \end{pmatrix}$$

where $A = (Z_n - Z_n^{-1})/2$ and $B = (Z_n + Z_n^{-1})/2$.

Define

$$f(U) = \mathrm{Tr}\left[\left((JM_k)^2 + \lambda^2 I_{2k}\right)^2\right] = 2\sum_{j=1}^{k}\left(\lambda_j^2 - \lambda^2\right)^2 \ .$$

Here, $\{\lambda_j\}_{j=1}^k$ are the symplectic eigenvalues of M_k and $\lambda = \mathrm{Tr}[B]/n$ is the averaged energy.

The calculation of $\mathbb{E}[f(U)]$ involves Weingarten calculus with $\mathcal{S}_4$.

2.7 Concentration

Concentration of symplectic eigenvalues:
For fixed k, suppose the energy of the initial pure Gaussian state is universally bounded; $\sup_n \|Z_n\|_\infty < \infty$ (n^z with $z < 1/8$ is possible). Then

$$\Pr\{f(U) > \epsilon\} < \exp(-c\epsilon n)$$

for some universal constant $c > 0$.

Concentration of entanglement entropy:
In addition, if $\lambda = \lambda_n$ is bounded below by $\mu > 1$, then

$$\Pr\{|S(M_k) - S(\lambda I)| > \epsilon\} < \exp\left(-c\frac{\epsilon^4}{\beta(\mu)^4}n\right)$$

for some universal constant $c > 0$; $S(\cdot)$ is von Neumann entropy, and

$$\beta(\mu) = \log\frac{\mu+1}{\mu-1} \ .$$

is called the inverse temperature.

Thank you very much.

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