Order unit norms arising from multi-qubit X-states

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References

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1. Separability, PPT, Bock-Positivity

In this talk, every state is assumed to be unnormalized.

A multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ is said to be *(fully) separable* if it is the sum of pure product states.

A multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ is said to be *entangled* if it is not separable.

For a given subset $S \subset \{1, 2, \cdots, n\}$, the partial transpose $T(S)$ on $\bigotimes_{i=1}^{n} M_{d_i}$ is a linear map satisfying

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^{T(S)} := b_1 \otimes b_2 \otimes \cdots \otimes b_n, \quad \text{with } b_i = \begin{cases} a_i^{\mathrm{t}}, & i \in S, \\ a_i, & i \notin S. \end{cases}$$

A multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ is called of *PPT* if $\varrho^{T(S)}$ is positive for every $S \subset \{1, 2, \cdots, n\}$.

It is obvious that every separable state is of PPT.

A self-adjoint matrix W is called block positive if $\langle W, \varrho\rangle \geqslant 0$ for all separable states ϱ.

Here, the bilinear pairing is given by

$$\langle W, \varrho\rangle = \mathrm{Tr}\,(\varrho W^{\mathrm{t}}) = \sum_{i,j} W_{i,j}\varrho_{i,j}.$$

In other words, the cone of block positive matrices is the dual cone of the cone of separable states.

Conversely, the cone of separable states is the dual cone of the cone of block positive matrices by Hahn-Banach type separation theorem.

Since the cone of states is self-dual, we have the inclusions:

$$
\begin{aligned}
&\text{the cone of separable states} \\
&\subset \text{the cone of states} \\
&\subset \text{the cone of block positive matrices.}
\end{aligned}
$$

2. X-states

A multi-qubit self-adjoint matrix $W \in \bigotimes_{i=1}^{n} M_{d_i}$ $(d_i = 2)$ is said to be X-shaped if it is of the form

$$W = \begin{pmatrix} s_{00\cdots00} & & & & & & & u_{00\cdots00} \\ & s_{00\cdots01} & & & & & u_{00\cdots01} & \\ & & \ddots & & & \iddots & & \\ & & & s_{01\cdots11} & u_{01\cdots11} & & & \\ & & & u_{10\cdots00} & s_{10\cdots00} & & & \\ & & \iddots & & & \ddots & & \\ & u_{11\cdots10} & & & & & s_{11\cdots10} & \\ u_{11\cdots11} & & & & & & & s_{11\cdots11} \end{pmatrix}.$$

We denote the above by $W = X(s, u)$ briefly.

A multi-qubit state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ $(d_i = 2)$ is said to be an X-state if it is X-shaped., that is, it is of the form

$$\varrho = \begin{pmatrix} a_{00\cdots00} & & & & & & & z_{00\cdots00} \\ & a_{00\cdots01} & & & & & z_{00\cdots01} & \\ & & \ddots & & & \iddots & & \\ & & & a_{01\cdots11} & z_{01\cdots11} & & & \\ & & & z_{10\cdots00} & a_{10\cdots00} & & & \\ & & \iddots & & & \ddots & & \\ & z_{11\cdots10} & & & & & a_{11\cdots10} & \\ z_{11\cdots11} & & & & & & & a_{11\cdots11} \end{pmatrix}.$$

A self-adjoint matrix $\varrho = X(a, z)$ is a state if and only if $\sqrt{a_{\mathbf{i}} a_{\bar{\mathbf{i}}}} \geqslant |z_{\mathbf{i}}|$ for all i.
Each partial transpose fixes diagonal entries and permutes anti-diagonal entries. An X-state $\varrho = X(a, z)$ is of PPT if and only if $\sqrt{a_i a_{\bar{\mathbf{i}}}} \geqslant |z_{\mathbf{j}}|$ for all $\mathbf{i}, \mathbf{j}$.

Theorem (HHK19)

(i) *The X-part of a multi-qubit separable state is again separable.*

(ii) *The X-part of a multi-qubit block positive matrix is again block positive.*

Theorem (HHK19)

(i) *An X-state is separable if and only if $\langle W, \varrho \rangle \geqslant 0$ for all X-shaped block positive W.*

(ii) *A self-adjoint X-shaped matrix is block positive if and only if $\langle W, \varrho \rangle \geqslant 0$ for all separable X-states ϱ.*

3. General Criterion

For an n-qubit X-shaped matrix $\mathrm{X}(s,u)$ with nonnegative diagonals, we consider the following two numbers:

$$\begin{aligned}
&\delta_n(s) \\
&= \inf\{s_{00\cdots00} r_1 r_2 \cdots r_{n-1} r_n + s_{00\cdots01} r_1 r_2 \cdots r_{n-1} r_n^{-1} + \cdots \\
&\qquad + s_{01\cdots11} r_1 r_2^{-1} \cdots r_{n-1}^{-1} r_n^{-1} + s_{10\cdots00} r_1^{-1} r_2 \cdots r_{n-1} r_n + \cdots \\
&\qquad + s_{11\cdots10} r_1^{-1} r_2^{-1} \cdots r_{n-1}^{-1} r_n + s_{11\cdots11} r_1^{-1} r_2^{-1} \cdots r_{n-1}^{-1} r_n^{-1} \\
&\qquad : r_i > 0\}
\end{aligned}$$

and

$$\begin{aligned}
&\|u\|_{\mathrm{X}_n} \\
&= \sup\{u_{00\cdots00} \alpha_1 \alpha_2 \cdots \alpha_{n-1} \alpha_n + u_{00\cdots01} \alpha_1 \alpha_2 \cdots \alpha_{n-1} \alpha_n^{-1} + \cdots \\
&\qquad + u_{01\cdots11} \alpha_1 \alpha_2^{-1} \cdots \alpha_{n-1}^{-1} \alpha_n^{-1} + u_{10\cdots00} \alpha_1^{-1} \alpha_2 \cdots \alpha_{n-1} \alpha_n + \cdots \\
&\qquad + u_{11\cdots10} \alpha_1^{-1} \alpha_2^{-1} \cdots \alpha_{n-1}^{-1} \alpha_n + u_{11\cdots11} \alpha_1^{-1} \alpha_2^{-1} \cdots \alpha_{n-1}^{-1} \alpha_n^{-1} \\
&\qquad : \alpha_i \in \mathbb{T}\}.
\end{aligned}$$

Theorem (HHK19)

An n-qubit X-shaped self-adjoint matrix $W = X(s,u)$ is block-positive if and only if the inequality

$$\delta_n(s) \geqslant \|u\|_{\mathsf{X}_n}$$

holds.

For an n-qubit X-state $\varrho = X(a,z)$, we also introduce the two numbers

$$\Delta_n(a) = \inf\{\langle s, a\rangle : \delta_n(s) = 1\},$$
$$\|z\|'_{\mathsf{X}_n} = \sup\{\langle u, z\rangle : \ \|u\|_{\mathsf{X}_n} = 1\},$$

which can be considered as duals of $\delta_n(s)$ and $\|u\|_{\mathsf{X}_n}$.

Theorem (HHK19)

An n-qubit X-state $\varrho = X(a,z)$ is separable if and only if the inequality

$$\Delta_n(a) \geqslant \|z\|'_{\mathsf{X}_n}$$

holds.

4. Computation to algebraic or analytic formulas

For a small n like $2, 3, 4$, we will use indices by natural numbers rather than multi-indices as following:

$$\mathrm{X}(a,b,z) = \begin{pmatrix} a_1 & & & & & & & z_1 \\ & a_2 & & & & & z_2 & \\ & & \ddots & & & \iddots & & \\ & & & a_8 & z_8 & & & \\ & & & \bar{z}_8 & b_8 & & & \\ & & \iddots & & & \ddots & & \\ & \bar{z}_2 & & & & & b_2 & \\ \bar{z}_1 & & & & & & & b_1 \end{pmatrix}.$$

$$a_1 = a_{0000}, a_2 = a_{0001}, \cdots, a_8 = a_{0111}$$
$$b_8 = a_{1000}, b_2 = a_{1110}, \cdots, b_1 = a_{1111}$$
$$z_1 = z_{0000}, z_2 = z_{0001}, \cdots, z_8 = z_{0111}$$
$$\bar{z}_8 = z_{1000}, \bar{z}_7 = z_{1110}, \cdots, \bar{z}_1 = z_{1111}.$$

I. Two qubit case

For $a, b \in \mathbb{R}^2_+$ and $z \in \mathbb{C}^2$, it is easy to check that

$$\Delta_2(a, b) = \min\{\sqrt{a_1 b_1}, \sqrt{a_2 b_2}\} \qquad \text{and} \qquad \|z\|'_{X_2} = \|z\|_\infty .$$

This is consistent with the fact that two qubit state is separable if and only if it is of PPT.

II. Three qubit case

Suppose that the X-part of a three qubit state ϱ is given by $X(a, b, z)$ with $a, b \in \mathbb{R}_+^4$ and $z \in \mathbb{C}^4$. In 2011, Gühne introduced the following numbers:

$$\begin{aligned}
\mathcal{L}(\varrho, u) :=& \mathrm{Re}\left(u_1 z_1 + u_2 z_2 + u_3 z_3 + u_4 \bar{z}_4\right), \qquad u \in \mathbb{C}^4 \\
\mathcal{F}(u) :=& \mathrm{Re}(u_1)\cos(\alpha+\beta+\gamma) - \mathrm{Im}(u_1)\sin(\alpha+\beta+\gamma) \\
&+ \mathrm{Re}(u_2)\cos(\alpha) - \mathrm{Im}(u_2)\sin(\alpha) \\
&+ \mathrm{Re}(u_3)\cos(\beta) - \mathrm{Im}(u_3)\sin(\beta) \\
&+ \mathrm{Re}(u_4)\cos(\gamma) - \mathrm{Im}(u_4)\sin(\gamma), \\
C(u) :=& \sup_{\alpha,\beta,\gamma} |\mathcal{F}(u)|.
\end{aligned}$$

(Gühne, 2011)
Suppose that the X-part of a three qubit state ϱ is given by $X(a, b, z)$. If ϱ is separable, then the inequality

$$\mathcal{L}(\varrho, u) \leqslant C(u)\ \Delta_\varrho$$

holds for every $u \in \mathbb{C}^4$, where the number Δ_ϱ is given by

$$\Delta_\varrho := \min\{\sqrt{a_1 b_1}, \sqrt{a_2 b_2}, \sqrt{a_3 b_3}, \sqrt{a_4 b_4}, \sqrt[4]{a_1 b_2 b_3 a_4}, \sqrt[4]{b_1 a_2 a_3 b_4}\}.$$

Theorem (CHK17/HHK19)

For a three qubit X*-state* $X(a,b,z)$, *we have*

$$\begin{aligned}&\Delta_3(a,b)\\ &=\min\{\sqrt{a_1b_1},\sqrt{a_2b_2},\sqrt{a_3b_3},\sqrt{a_4b_4},\sqrt[4]{a_1b_2b_3a_4},\sqrt[4]{b_1a_2a_3b_4}\}.\end{aligned}$$

and

$$\|z\|'_{\mathrm{X}_3}=\sup\{\frac{\mathcal{L}(\varrho,u)}{C(u)}:u\in\mathbb{C}^4\}.$$

Let

$$\phi = (\theta_1 + \theta_4) - (\theta_2 + \theta_3).$$

for $z_i = r_i e^{\mathrm{i}\theta_i}$.

Theorem (HK17-2)

For $z \in \mathbb{C}_+^4$ with $|z_i| = r$, we have

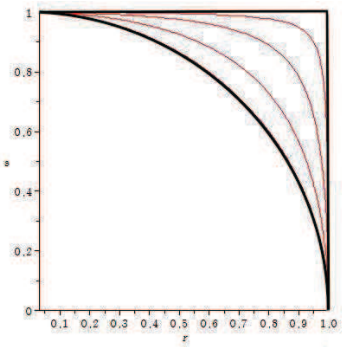
$$\|z\|'_{X_3} = r\sqrt{1 + |\sin(\phi/2)|}.$$

Theorem (CHK17)

For $z_i \in \mathbb{C}^4$ *with* $|z_{i_1}| = |z_{i_2}| = r$ *and* $|z_{i_3}| = |z_{i_4}| = s$*, we have*

$$\|z\|'_{\mathsf{X}} = \sqrt{\frac{r^2 t_0^2 + 2rst_0|\sin(\phi/2)| + s^2}{t_0^2 + 1}},$$

where $t_0 = \dfrac{r^2 - s^2 + \sqrt{(r^2 - s^2)^2 + (2rs\sin(\phi/2))^2}}{2rs|\sin(\phi/2)|}$.



Two thick curves, circle and rectangle, represent the boundary of the separability region (r, s) of the state

$$X((1,1,1,1),(1,1,1,1),(e^{i\theta}r, r, s, s))$$

for $\theta = \pi$ and $\theta = 0$, respectively. The other curves represent the separability regions for $\theta = \pi/2, \pi/4, \pi/10$.

Theorem (CHK17)

For $z \in \mathbb{C}^4$, we have

$$\|z\|'_{X_3} \geqslant \frac{\sqrt{2}m_1 m_2}{\sqrt{m_1^2 + m_2^2}} \sqrt{1 + |\sin(\phi/2)|}$$

for $m_1 = \frac{|z_{i_1}|+|z_{i_2}|}{2}$ and $m_2 = \frac{|z_{i_3}|+|z_{i_4}|}{2}$.

III. GHZ diagonal case

A multi-qubit X-state $\mathrm{X}(a, z)$ is GHZ diagonal if and only if $a_{\mathbf{i}} = a_{\bar{\mathbf{i}}}$ and $z_{\mathbf{i}} \in \mathbb{R}$.

Theorem (HK17-1/HHK19)

For $a \in \mathbb{R}_+^{2^n}$, we have

$$\Delta_n(a) = \min_{\mathbf{i}} a_{\mathbf{i}}.$$

For a three qubit GHZ diagonal state $\mathrm{X}(a, a, z)$, we define the real numbers

$$\begin{aligned}
\lambda_5 &= 2(+z_1 + z_2 + z_3 + z_4), \quad & t_1 &= z_1(-z_1^2 + z_2^2 + z_3^2 + z_4^2) - 2z_2 z_3 z_4, \\
\lambda_6 &= 2(-z_1 - z_2 + z_3 + z_4), \quad & t_2 &= z_2(+z_1^2 - z_2^2 + z_3^2 + z_4^2) - 2z_1 z_3 z_4, \\
\lambda_7 &= 2(-z_1 + z_2 - z_3 + z_4), \quad & t_3 &= z_3(+z_1^2 + z_2^2 - z_3^2 + z_4^2) - 2z_1 z_2 z_4, \\
\lambda_8 &= 2(-z_1 + z_2 + z_3 - z_4), \quad & t_4 &= z_4(+z_1^2 + z_2^2 + z_3^2 - z_4^2) - 2z_1 z_2 z_3.
\end{aligned}$$

and the condition

$$(*) \qquad t_1 t_4 \lambda_6 \lambda_7 < 0 \quad \text{and} \quad t_2 t_3 \lambda_5 \lambda_8 > 0.$$

Theorem (HK17-1)

For $z \in \mathbb{R}^4$, we have the following:

(i) *if* $\lambda_5\lambda_6\lambda_7\lambda_8 \leqslant 0$, *then* $\|z\|'_{X_3} = \|z\|_\infty$;

(ii) *if* $\lambda_5\lambda_6\lambda_7\lambda_8 > 0$ *and the condition* $(*)$ *does not hold, then* $\|z\|'_{X_3} = \|z\|_\infty$;

(iii) *if* $\lambda_5\lambda_6\lambda_7\lambda_8 > 0$ *and the condition* $(*)$ *holds, then*

$$\|z\|'_{X_3} = \frac{\sqrt{(\lambda_5\lambda_6 + \lambda_7\lambda_8)(\lambda_5\lambda_7 + \lambda_6\lambda_8)(\lambda_5\lambda_8 + \lambda_6\lambda_7)}}{8\sqrt{\lambda_5\lambda_6\lambda_7\lambda_8}}$$

IV. Four qubit case

For $1 = 00, 2 = 01, 3 = 10, 4 = 11$, we observe that

$$(0,0) + (1,1) = (0,1) + (1,0).$$

We denote this relation by

$$1 + 4 \equiv 2 + 3.$$

The quantity

$$\begin{aligned}&\Delta_3(a,b)\\ &=\min\{\sqrt{a_1b_1},\sqrt{a_2b_2},\sqrt{a_3b_3},\sqrt{a_4b_4},\sqrt[4]{a_1b_2b_3a_4},\sqrt[4]{b_1a_2a_3b_4}\}.\end{aligned}$$

matches to

$$1\equiv 1,\ 2\equiv 2,\ 3\equiv 3,\ 4\equiv 4,\ 1+4\equiv 2+3.$$

The quantity

$$\phi=(\theta_1+\theta_4)-(\theta_2+\theta_3).$$

matches to

$$1+4\equiv 2+3.$$

For $1 = 000, 2 = 001, 3 = 010, 4 = 011, 5 = 100, 6 = 101, 7 = 110, 8 = 111$, all nontrivial relations are

$$1 \equiv 1,\ 2 \equiv 2,\ 3 \equiv 3,\ 4 \equiv 4,\ 5 \equiv 5,\ 6 \equiv 6,\ 7 \equiv 7,\ 8 \equiv 8$$

and

$$\begin{gathered}
1+8 \equiv 2+7 \equiv 3+6 \equiv 4+5, \\
1+4 \equiv 2+3, \quad 5+8 \equiv 6+7, \\
1+6 \equiv 2+5, \quad 3+8 \equiv 4+7, \\
1+7 \equiv 3+5, \quad 2+8 \equiv 4+6.
\end{gathered}$$

and

$$\begin{array}{ll}
1+1+8 \equiv 2+3+5, & 2+2+7 \equiv 1+4+6, \\
3+3+6 \equiv 1+4+7, & 4+4+5 \equiv 2+3+8, \\
4+5+5 \equiv 1+6+7, & 3+6+6 \equiv 2+5+8, \\
2+7+7 \equiv 3+5+8, & 1+8+8 \equiv 4+6+7.
\end{array}$$

Theorem (H)

For $a, b \in \mathbb{R}_+^8$, we have

$$\begin{aligned}\Delta_4(a,b) = \min \Big\{ \sqrt{a_{\mathbf{i}} b_{\mathbf{i}}} : 1 \leqslant \mathbf{i} \leqslant 8 \Big\} \\ \cup \left\{ \sqrt[4]{a_{\mathbf{i}_1} a_{\mathbf{i}_2} b_{\mathbf{j}_1} b_{\mathbf{j}_2}} : \mathbf{i}_1 + \mathbf{i}_2 \equiv \mathbf{j}_1 + \mathbf{j}_2 \right\} \\ \cup \left\{ \sqrt[6]{a_{\mathbf{i}_1} a_{\mathbf{i}_2} a_{\mathbf{i}_3} b_{\mathbf{j}_1} b_{\mathbf{j}_2} b_{\mathbf{j}_3}} : \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 \equiv \mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3 \right\}.\end{aligned}$$

For example, $1 + 1 + 8 \equiv 2 + 3 + 5$
$((0,0,0) + (0,0,0) + (1,1,1) = (0,0,1) + (0,1,0) + (1,0,0))$
yields $\sqrt[6]{a_1^2 a_8 b_2 b_3 b_5}$ and $\sqrt[6]{b_1^2 b_8 a_2 a_3 a_5}$.
The set which we take a minimum contains $8 + 24 + 16 = 48$ algebraic elements.

We define

$$\phi_1 := \theta_1 + \theta_4 - \theta_2 - \theta_3, \qquad \phi_2 := \theta_5 + \theta_8 - \theta_6 - \theta_7,$$
$$\phi_3 := \theta_1 + \theta_8 - \theta_4 - \theta_5, \qquad \phi_4 := \theta_2 + \theta_7 - \theta_3 - \theta_6.$$

These match to

$$1 + 4 \equiv 2 + 3, \quad 5 + 8 \equiv 6 + 7, \quad 1 + 8 \equiv 4 + 5, \quad 2 + 7 \equiv 3 + 6.$$

Let $z \in \mathbb{C}^4$ with $z_i = r_i e^{\mathrm{i}\theta_i}$ and $\phi = (\theta_1 + \theta_4) - (\theta_2 + \theta_3)$.

We have

$$\|c\|'_{X_3} = \|(c_1, c_1, c_2, c_2, c_3, c_3, c_4, c_4)\|'_{X_4}.$$

with

$$\phi_1 = \phi_2 = 0, \qquad \phi_3 = \phi_4 = \phi.$$

We also have

$$\|c\|'_{X_3} = \|(c_1, c_2, c_3, c_4, c_1, c_2, c_3, c_4)\|'_{X_4}.$$

with

$$\phi_1 = \phi_2 = \phi, \qquad \phi_3 = \phi_4 = 0.$$

Theorem (H)

For $z \in \mathbb{C}^8$ with $|z_i| = r$, we have the following:

(i) *if $\phi_3 = \phi_4 = 0$, then*

$$\begin{aligned}&\|z\|'_{\mathsf{X}_4}\\ =& r\left(\max\{|\cos(\phi_1/4)|, |\cos(\phi_2/4)|\} + \max\{|\sin(\phi_1/4)|, |\sin(\phi_2/4)|\}\right)\end{aligned}$$

(ii) *if $\phi_1 = \phi_2 = 0$, then*

$$\begin{aligned}&\|z\|'_{\mathsf{X}_4}\\ =& r\left(\max\{|\cos(\phi_3/4)|, |\cos(\phi_4/4)|\} + \max\{|\sin(\phi_3/4)|, |\sin(\phi_4/4)|\}\right)\end{aligned}$$

This recovers the three qubit result

$$r\left(|\cos(\phi/4)| + |\sin(\phi/4)|\right) = r\sqrt{1 + |\sin(\phi/2)|}.$$

Theorem (H)

For $z \in \mathbb{C}^8$ *with* $|z_i| = r$ *and* $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi$*, we have*

$$\|z\|'_{\mathsf{X}_4} = r(1 + |\sin(\phi/2)|).$$

Theorem (H)

Suppose that $z \in \mathbb{C}^8$ *and* $|z_i| = r_i$. *Then,*

(i) *we have*

$$
\begin{aligned}
&\|z\|'_{X_4} \\
&\geqslant \min\{m_1, m_2\} \frac{\max\{|\cos(\phi_1/4)|, |\cos(\phi_2/4)|\} + \max\{|\sin(\phi_1/4)|, |\sin(\phi_2/4)|\}}{\max\{|\cos(\phi_3/4)|, |\cos(\phi_4/4)|\} + \max\{|\sin(\phi_3/4)|, |\sin(\phi_4/4)|\}}
\end{aligned}
$$

for $m_1 = (r_1 + r_2 + r_3 + r_4)/4$ *and* $m_2 = (r_5 + r_6 + r_7 + r_8)/4$*;*

(ii) *we have*

$$
\begin{aligned}
&\|z\|'_{X_4} \\
&\geqslant \min\{m_3, m_4\} \frac{\max\{|\cos(\phi_3/4)|, |\cos(\phi_4/4)|\} + \max\{|\sin(\phi_3/4)|, |\sin(\phi_4/4)|\}}{\max\{|\cos(\phi_1/4)|, |\cos(\phi_2/4)|\} + \max\{|\sin(\phi_1/4)|, |\sin(\phi_2/4)|\}}
\end{aligned}
$$

for $m_3 = (r_1 + r_4 + r_5 + r_8)/4$ *and* $m_4 = (r_2 + r_3 + r_6 + r_7)/4$*.*

IV. *n*-qubit case

By the nontrivial relations

$$\mathbf{i}_1 + \mathbf{i}_2 + \cdots + \mathbf{i}_m \equiv \mathbf{j}_1 + \mathbf{j}_2 + \cdots + \mathbf{j}_m$$

we can define $\tilde{\Delta}_n(a)$ for $a \in \mathbb{R}_+^{2^n}$.

What is the formal definition of *nontrivial*?

For $1 + 4 \equiv 2 + 3$, we observe that

$$\{000, 011, \overline{001}, \overline{010}\} = \{000, 011, 110, 101\}$$

and

$$\{\color{blue}{0}00, \color{blue}{0}11, \color{red}{1}10, \color{red}{1}01\}$$
$$\{0\color{blue}{0}0, 0\color{red}{1}1, 1\color{red}{1}0, 0\color{blue}{0}1\}$$
$$\{00\color{blue}{0}, 01\color{red}{1}, 11\color{blue}{0}, 00\color{red}{1}\}$$

A multiset is a collection which allows repetition of elements, unlike a set.

A multiset T of length n indices of $0, 1$ will be said to be *balanced* if the number of indices $\mathbf{i}$ in T with $\mathbf{i}(k) = 0$ coincides with the number of indices $\mathbf{i} \in T$ with $\mathbf{i}(k) = 1$ for every $k = 1, 2, \ldots, n$.

We say that a balanced multiset is *irreducible* when it cannot be partitioned into balanced multisets.

It is easily seen that the cardinality of a balanced multiset must be even and the set $\mathcal{G}_n$ of all irreducible balanced multisets of length n indices of $0, 1$ is finite.

For $a \in \mathbb{R}^{2^n}$, we define

$$\tilde{\Delta}_n(a) := \min \left\{ \left(\prod_{\mathbf{i} \in \mathbf{T}} a_{\mathbf{i}} \right)^{1/\#(T)} : T \in \mathcal{G}_n \right\}$$

Theorem (HHK19)

For $a \in \mathbb{R}^{2^n}$, we have

$$\tilde{\Delta}_n(a) \geqslant \Delta_n(a).$$

5. Function systems and order unit norms

A unital subspace of $C(K)$ had been abstractly characterized by Kadison. It is called a function system.

Let V be an ordered vector space with a distinguished element I.

1. We call $I \in V$ an order unit for V provided that for every $x \in V$, there exists a positive real r such that $rI + x \in V^+$.
2. We call I Archimedean if $\varepsilon I + x \in V^+$ for all $\varepsilon > 0$ implies that $x \in V^+$.
3. V is called a function system if it has an Archimedean order unit.

A function system V has a canonical norm

$$\|x\| := \inf\{r > 0 : -rI \leqslant x \leqslant rI\},$$

which is called an order unit norm of V.

(Kadison,1951)
Let V be a function system. There exists a unital order embedding

$$\Phi : V \hookrightarrow C(K)$$

for a compact Hausdorff space K.

This embedding is isometric with respect to the order unit norm.

Functions systems and compact convex sets are dual to each other.

For a function system V, the state space $S(V)$ is a compact convex set.

For a compact convex set K, the space $\mathrm{Aff}(K)$ of affine functions on K is a function system.
Moreover, we have

$$\mathrm{Aff}(S(V)) \simeq V \qquad \text{and} \qquad S(\mathrm{Aff}(K)) \simeq K.$$

For function systems V and W, we define

$$V^+ \otimes W^+ := \{\sum_{i=1}^{n} a_i \otimes b_i \in V \otimes W : n \in \mathbb{N}, a_i \in V^+, b_i \in W^+\}$$

and

$$(V \otimes_{\max} W)^+ := \{z \in V \otimes W : \forall \varepsilon > 0, z + \varepsilon 1_V \otimes 1_V \in V^+ \otimes W^+\}.$$

The maximal tensor product

$$V \otimes_{\max} W = (V \otimes W, (V \otimes_{\max} W)^+, 1_V \otimes 1_W)$$

is a function system.

For function systems V and W, we define

$$(V \otimes_{\min} W)^+ = \{z \in V \otimes W : (f \otimes g)(z) \geqslant 0, f \in S(V), g \in S(W)\}$$

The minimal tensor product

$$V \otimes_{\min} W = (V \otimes W, (V \otimes_{\min} W)^+, 1_V \otimes 1_W)$$

is a function system.

$(M_m \otimes_{\max} M_n)^+$ coincides with the cone of separable states.
(We do not need $\varepsilon > 0$ by the Carathéodory theorem.)

$(M_m \otimes_{\min} M_n)^+$ coincides with the cone of block positive matrices.

$\frac{1}{2^n}\|u\|_{\mathsf{X}_n}$ coincides with the order unit norm of self-adjoint anti-diagonal matrix $\mathsf{X}(0, u)$ in $\otimes^n_{\min} M_2$.

$\|z\|'_{\mathsf{X}_n}$ coincides with the order unit norm of self-adjoint anti-diagonal matrix $\mathsf{X}(0, z)$ in $\otimes^n_{\max} M_2$.

Let $s_{\mathbf{i}}, a_{\mathbf{i}} > 0$ and $\delta_n(s) = 2^n$, $\Delta_n(a) = 1$.

Two order units I and $\mathrm{X}(s, 0)$ determine the same order unit norm on the self-adjoint anti-diagonal matrices $\mathrm{X}(0, u)$ in $\otimes_{\min}^n M_2$.

Two order units I and $\mathrm{X}(a, 0)$ determine the same order unit norm on the self-adjoint anti-diagonal matrices $\mathrm{X}(0, z)$ in $\otimes_{\max}^n M_2$.