Asymptotic decoupling property, mixing condition and Hidden Markovian Process in quantum system

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- Asymptotic decoupling property and mixing condition for CP map
- Markovian process and Hidden Markovian process
- Quantum Markovian process
- Quantum Hidden Markovian process
- (Finite-length) analysis for classical Markovian process
- (Finite-length) analysis for quantum hidden Markovian process
- Conclusion

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Ergodic, Mixing, and Asymptotic Decoupling conditions for TP-CP Γ

- Ergodic $\displaystyle \frac{1}{n}\sum_{k=0}^{n}\Gamma^{k}(\rho)\rightarrow\exists\rho_0$

$\Leftrightarrow$ $\dim\mathrm{Ker}(\Gamma-\iota)=1$

ι : identity map

- Mixing $\Gamma^{n}(\rho)\rightarrow\exists\rho_0$

- Asymptotic decoupling

$$\left\|\Gamma^{n}(\rho)-\mathrm{Tr}_B\Gamma^{n}(\rho)\otimes\mathrm{Tr}_A\Gamma^{n}(\rho)\right\|_1\rightarrow 0$$

Equivalence relations

- Γ is mxing.
- $\Gamma^{\otimes 2}$ is ergodic.
- $\dim \mathrm{Ker}(\Gamma^{\otimes 2} - \iota) = 1$
- $\Gamma^{\otimes 2}$ is asymptotic decoupling.
- $\Gamma \otimes \iota$ is asymptotic decoupling.

Irreducible (equivalence relations)

- Γ is ergodic and $\rho_0 > 0$.
- $\Gamma(\rho) \leq \alpha\rho \Rightarrow \rho > 0$.
- $\exp(t\Gamma(\rho)) > 0$.
- $(\imath + \Gamma)^{(\dim \mathcal{H})^2 - 1}(\rho) > 0$.

Classical case (W :transition matrix)

For any x, x', there exists n such that

$$W^n(x \mid x') > 0$$

Primitive (equivalence relations)

- Γ is mixing and irreducible.
- Γ is mixing and $\rho_0 > 0$.
- $\Gamma^{\otimes 2}$ is ergodic and $\rho_0 > 0$
- $\Gamma^{\otimes 2}(\rho) \leq \alpha\rho \Rightarrow \rho > 0.$
- $\exp(t\Gamma^{\otimes 2}(\rho)) > 0.$
- $(\imath + \Gamma^{\otimes 2})^{(\dim \mathcal{H})^4 - 1}(\rho) > 0.$

Ergodic, Mixing, and Asymptotic Decoupling conditions for general CP Γ

- Ergodic $\dfrac{1}{n}\sum_{k=0}^{n}\Gamma^k(\rho)/r(\Gamma)^k \rightarrow (\mathrm{Tr}A_0\rho)\rho_0$

$r(\Gamma)$: spectral radius

$$\Leftrightarrow \quad \dim \mathrm{Ker}(\Gamma - r(\Gamma)\iota) = 1$$

- Mixing $\Gamma^n(\rho)/r(\Gamma)^n \rightarrow (\mathrm{Tr}A_0\rho)\rho_0$

- Asymptotic decoupling

$$\left\| \Gamma^n(\rho) - \mathrm{Tr}_B\Gamma^n(\rho) \otimes \mathrm{Tr}_A\Gamma^n(\rho) \right\|_1 \rightarrow 0$$

Irreducible (general case) (equivalence relations)

- Γ is ergodic and $\rho_0 > \mathbf{0}, A_0 > \mathbf{0}$.
- $\Gamma(\rho) \leq \alpha\rho \Rightarrow \rho > \mathbf{0}.$
- $\mathbf{exp}(t\Gamma(\rho)) > \mathbf{0}.$
- $(\iota + \Gamma)^{(\mathbf{dim}\,\mathcal{H})^2\,\mathbf{-1}}(\rho) > \mathbf{0}.$

Primitive (general case) (equivalence relations)

- Γ is mixing and irreducible.
- Γ is mixing and $\rho_0 > 0, A_0 > 0$.
- $\Gamma^{\otimes 2}$ is ergodic and $\rho_0 > 0, A_0 > 0$.
- $\Gamma^{\otimes 2}(\rho) \leq \alpha\rho \Rightarrow \rho > 0.$
- $\exp(t\Gamma^{\otimes 2}(\rho)) > 0.$
- $(\iota + \Gamma^{\otimes 2})^{(\dim \mathcal{H})^4 - 1}(\rho) > 0.$

Theorem

$\sum_i \Gamma_i$:ergodic $\Leftrightarrow$ $\sum_i r_i \Gamma_i$:ergodic

$r_i > 0$

$\sum_i \Gamma_i$:mixing $\Leftrightarrow$ $\sum_i r_i \Gamma_i$:mixing

$r_i > 0$

$\sum_i \Gamma_i$:irreducible $\Leftrightarrow$ $\sum_i r_i \Gamma_i$:irreducible

$r_i > 0$

$\sum_i \Gamma_i$:primitive $\Leftrightarrow$ $\sum_i r_i \Gamma_i$:primitive

$r_i > 0$

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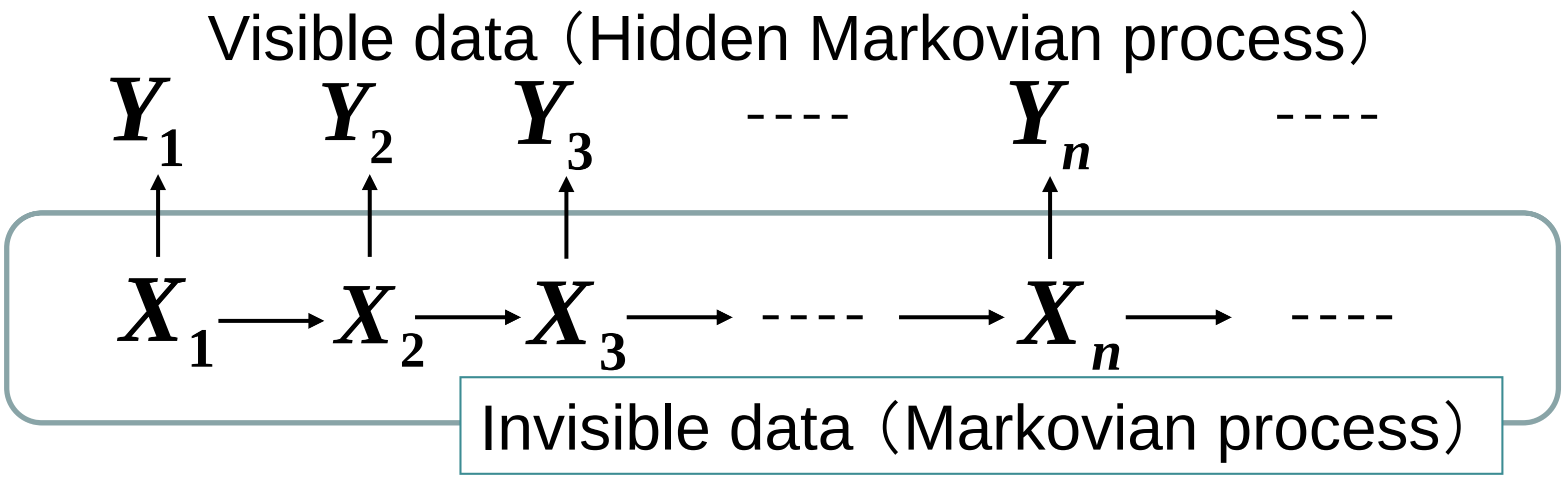
Markovian process and Hidden Markovian process



$P_{X_{i+1}Y_i|X_i}$:Transition matrix (Hidden Markovian process)

$P_{X_{i+1}|X_i}$:Transition matrix (Markovian process)

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Quantum Markovian process

$$\rho_1 \xrightarrow{\Gamma} \rho_2 \xrightarrow{\Gamma} \rho_3 \xrightarrow{\Gamma} \text{----} \xrightarrow{\Gamma} \rho_n \xrightarrow{\Gamma} \text{----}$$

Γ:TP-CP map

$$\rho_{i+1} = \Gamma(\rho_i)$$

This model has the following problems.

(1)There is no measurement. We obtain no Information from the system.
(2) Once the state ρ_{i+1} is generated, the previous state ρ_i has been lost.

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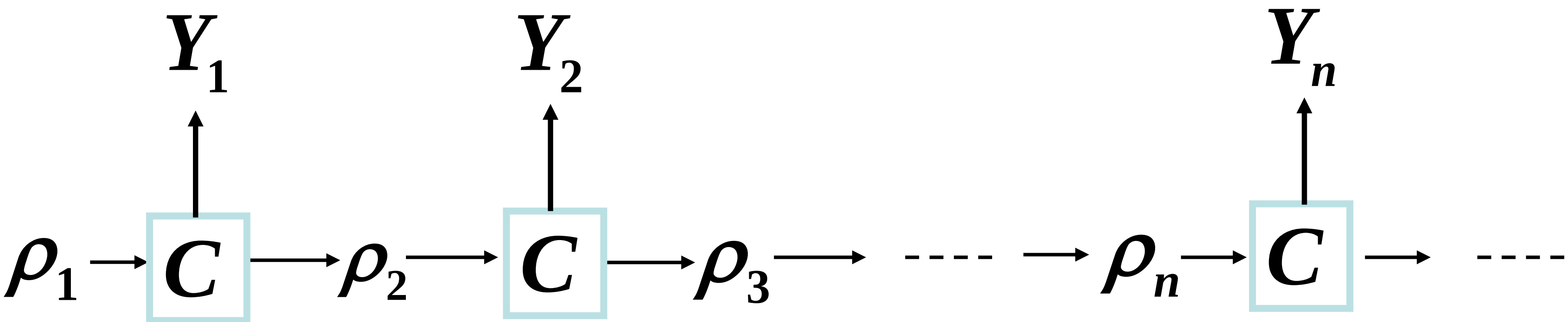
Quantum Sequential Measurement =Quantum Hidden Markovian process



$C = \{C_\omega\}_\omega$; CP-map valued measure(Instrument)

If sequential measurements have been done, we obtain quantum hidden Markovian process.

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(Finite-length) analysis for classical Markovian process

Various analyses were done with cumulant generating function.

Watanabe-MH AAP2017

$$X^n := X_1 + \cdots + X_n$$

$$\phi_n(\theta) := \log E[e^{\theta X^n}]$$

$$W(\theta)_{x,x'} := e^{\theta X(x)} P_{X_{i+1}|X_i}(x|x')$$

$\lambda(\theta)$: Perron Frobenius eigenvalue of $W(\theta)$

$$\phi(\theta) := \log \lambda(\theta)$$

$$n\phi(\theta) + \underline{\delta}(\theta) \le \phi_n(\theta) \le n\phi(\theta) + \overline{\delta}(\theta)$$

(Finite-length) analysis for classical Markovian process

$$n\phi(\theta)+\underline{\delta}(\theta)\le \phi_n(\theta)\le n\phi(\theta)+\overline{\delta}(\theta)$$

Watanabe-MH AAP2017

Markov version of central limit theorem
Finite-length analysis for tail probability
Large deviation analysis
Moderate deviation analysis

Central limit theorem

$$\Pr\left\{\frac{X^n - n\phi'(0)}{\sqrt{\phi''(0)n}} < R\right\} \to \int_{-\infty}^{R} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$$

Large deviation

$$-\frac{1}{n}\log\Pr\left\{X^n > nR\right\} \to \sup_{\theta\geq 0}\theta R - \phi(\theta)$$

Moderate deviation

$$-\frac{1}{n^{1-2t}}\log\Pr\left\{X^n > n\phi'(0) + n^{1-t}\delta\right\} \to \frac{\delta^2}{2\phi''(0)}$$

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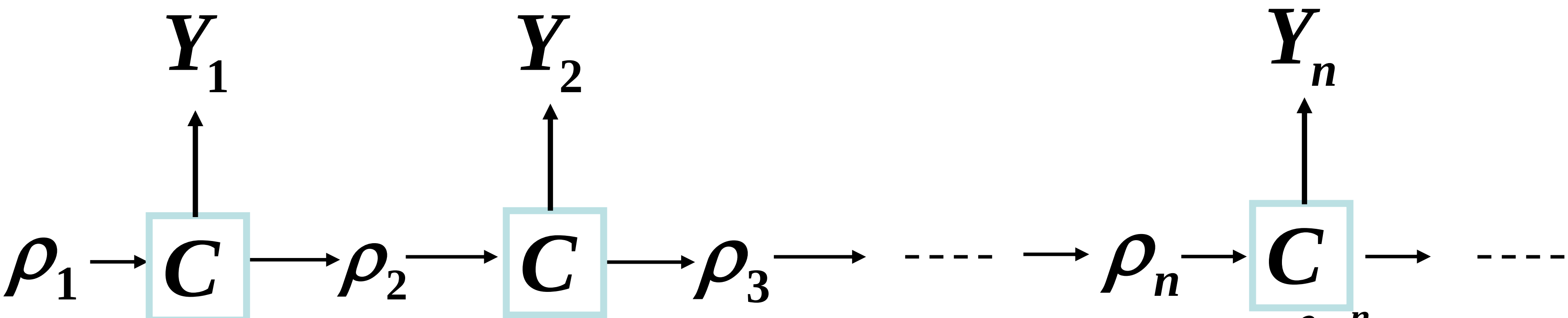
(Finite-length) analysis for quantum hidden Markovian process



$$Y^n := Y_1 + \cdots + Y_n, \quad \phi_n(\theta) := \log E[e^{\theta Y^n}]$$

$$W_\theta : \rho \mapsto \sum_{\omega \in \Omega} e^{\theta Y(\omega)} C_\omega(\rho) \qquad \phi(\theta) := \log \lambda(\theta)$$

$\lambda(\theta):$ Spectrum radius of W_θ

We employ Perron Frobenius theorem with generalized cone.

(Finite-length) analysis for quantum hidden Markovian process

$$n\phi(\theta)+\underline{\delta}(\theta)\le \phi_n(\theta)\le n\phi(\theta)+\overline{\delta}(\theta)$$

Markov version of central limit theorem
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Central limit theorem

$$\Pr\left\{\frac{Y^n - n\phi'(0)}{\sqrt{\phi''(0)n}} < R\right\} \to \int_{-\infty}^{R} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$$

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Conclusion

- We have discussed properties (ergodicity, mixing, irreducible, primitive, and asymptotic decoupling) for a dynamical map on quantum system.
- We have derived several equivalence relations.

Conclusion

- We have overseen the analogy between classical Markovian process and quantum hidden Markovian process.
- Using this analogy, we have derived the following for quantum hidden Markovian process.
  - Markov version of central limit theorem
  - Finite-length analysis for tail probability
  - Large deviation analysis
  - Moderate deviation analysis

Important information

- All the obtained results can be extended to general probabilistic theory.

References

- S. Watanabe and M. Hayashi, “Finite-length Analysis on Tail probability for Markov Chain and Application to Simple Hypothesis Testing,” *Annals of Applied Probability*, **27**(2) 811-845 (2017).
- R. Schrader, “Perron-Frobenius Theory for Positive Maps on Trace Ideals,” arXiv math-ph/0007020(2000).
- MH and Y. Yoshida, “Asymptotic and non-asymptotic analysis for a hidden Markovian process with a quantum hidden system,” *J. Phys. A,* **51**(33) 335304 (2018).
- Y. Yoshida, MH, “ Mixing and Asymptotic Decoupling Properties in General Probabilistic Theory”, Arxiv: 1801.03988.
- D. Burgarth, G. Chiribella, V. Giovannetti, P. Perinotti, and K. Yuasa,“Ergodic and mixing quantum channels in finite dimensions,” *New J. Phys.* **15**(7) 073045 (2013).

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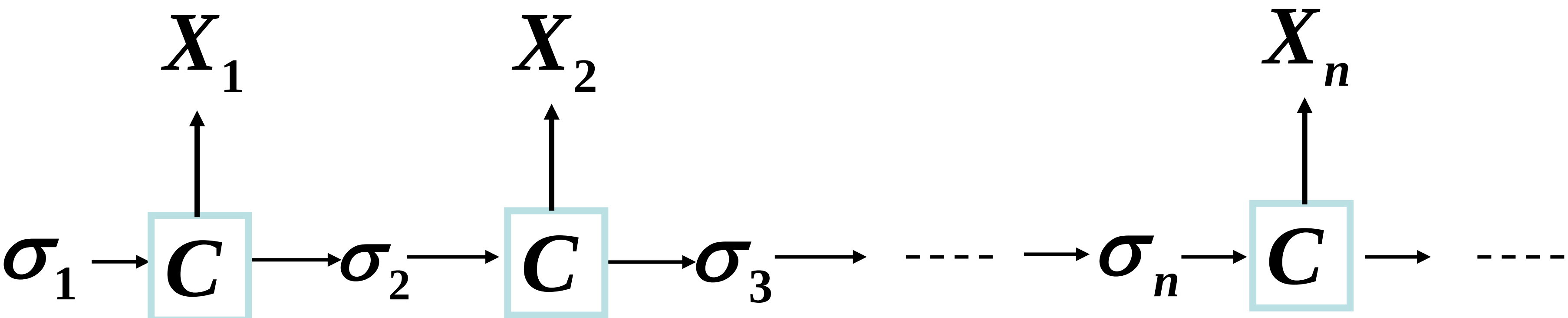

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If sequential measurements have been done, we obtain quantum hidden Markovian process.