MAQIT 2019, SNU, Korea

Noise-disturbance relation and the Galois connection of quantum measurements

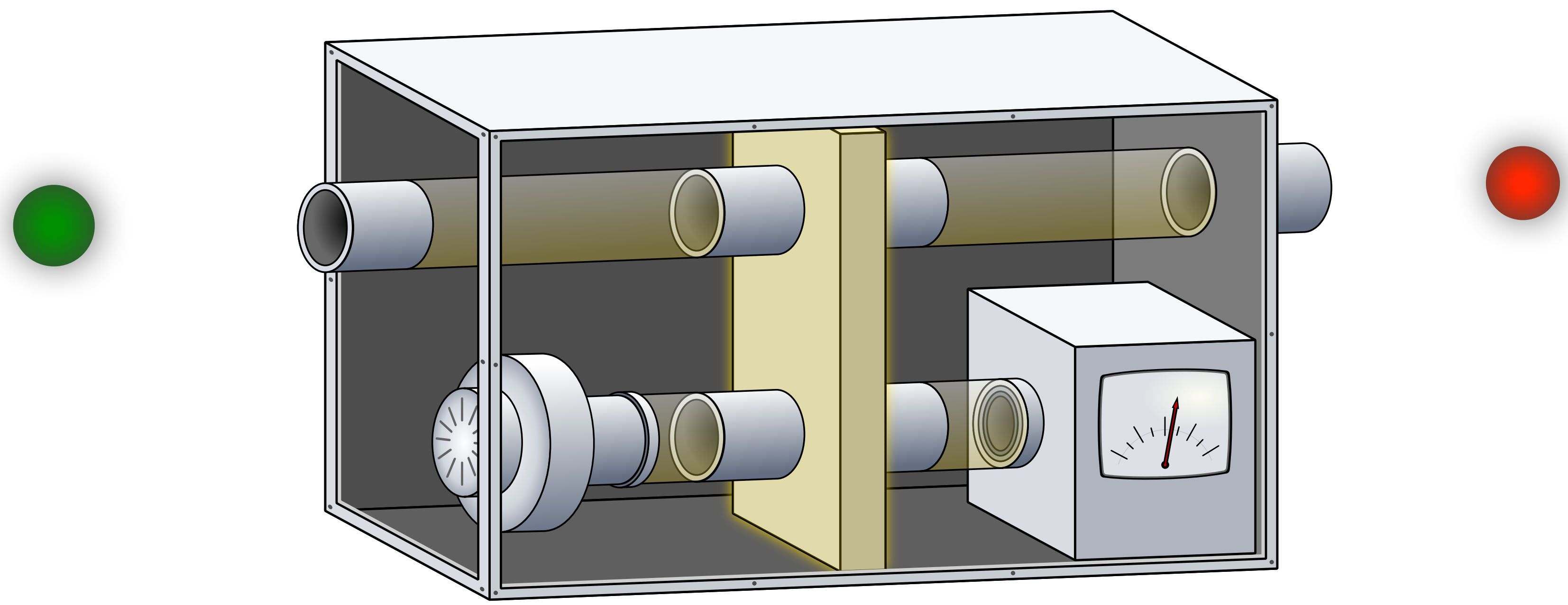
Teiko Heinosaari

University of Turku
Finland

Turku Quantum Technology

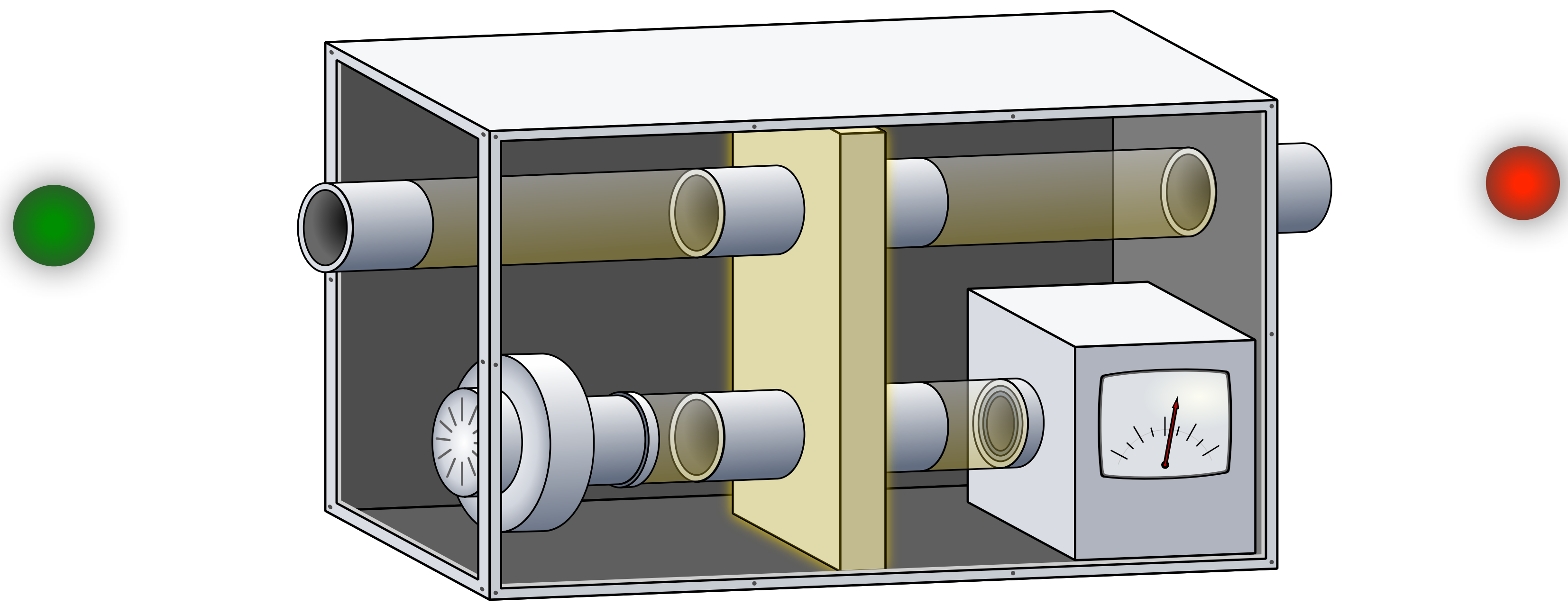
Quantum Technology Finland

noise-disturbance tradeoff



$$noise \cdot disturbance \geq 1$$

noise-disturbance tradeoff

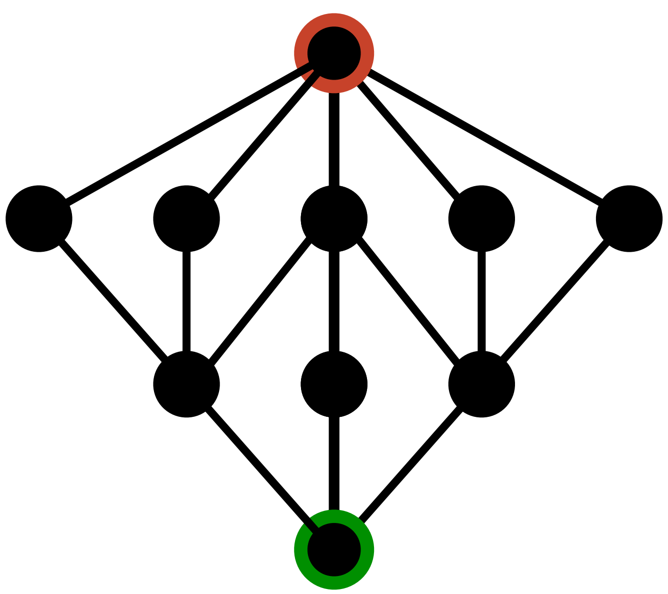


$$noise \cdot disturbance \geq 1$$

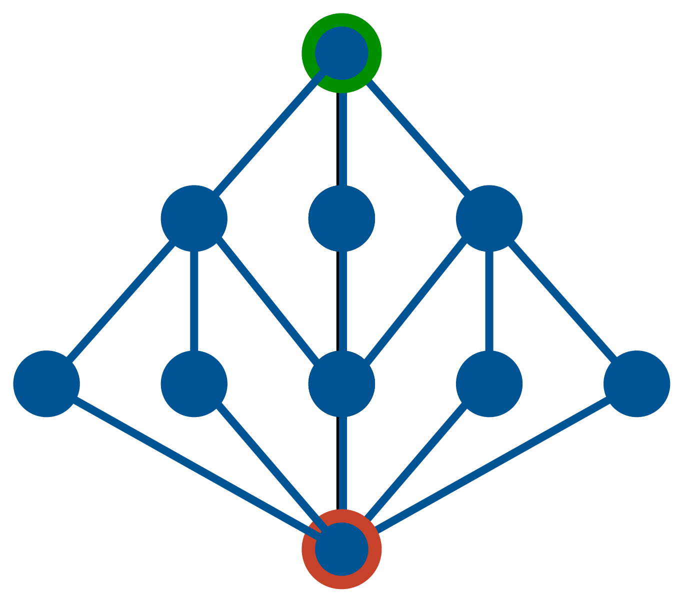
structural relation?

Galois theory

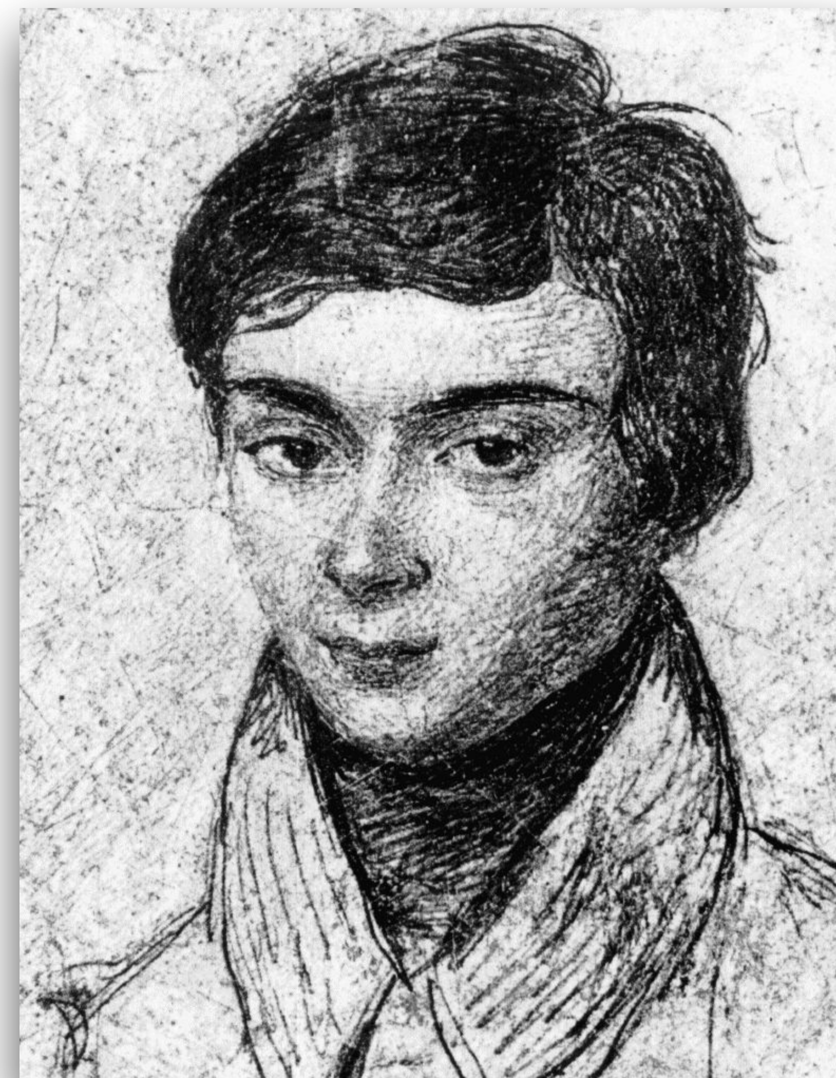
fields



groups

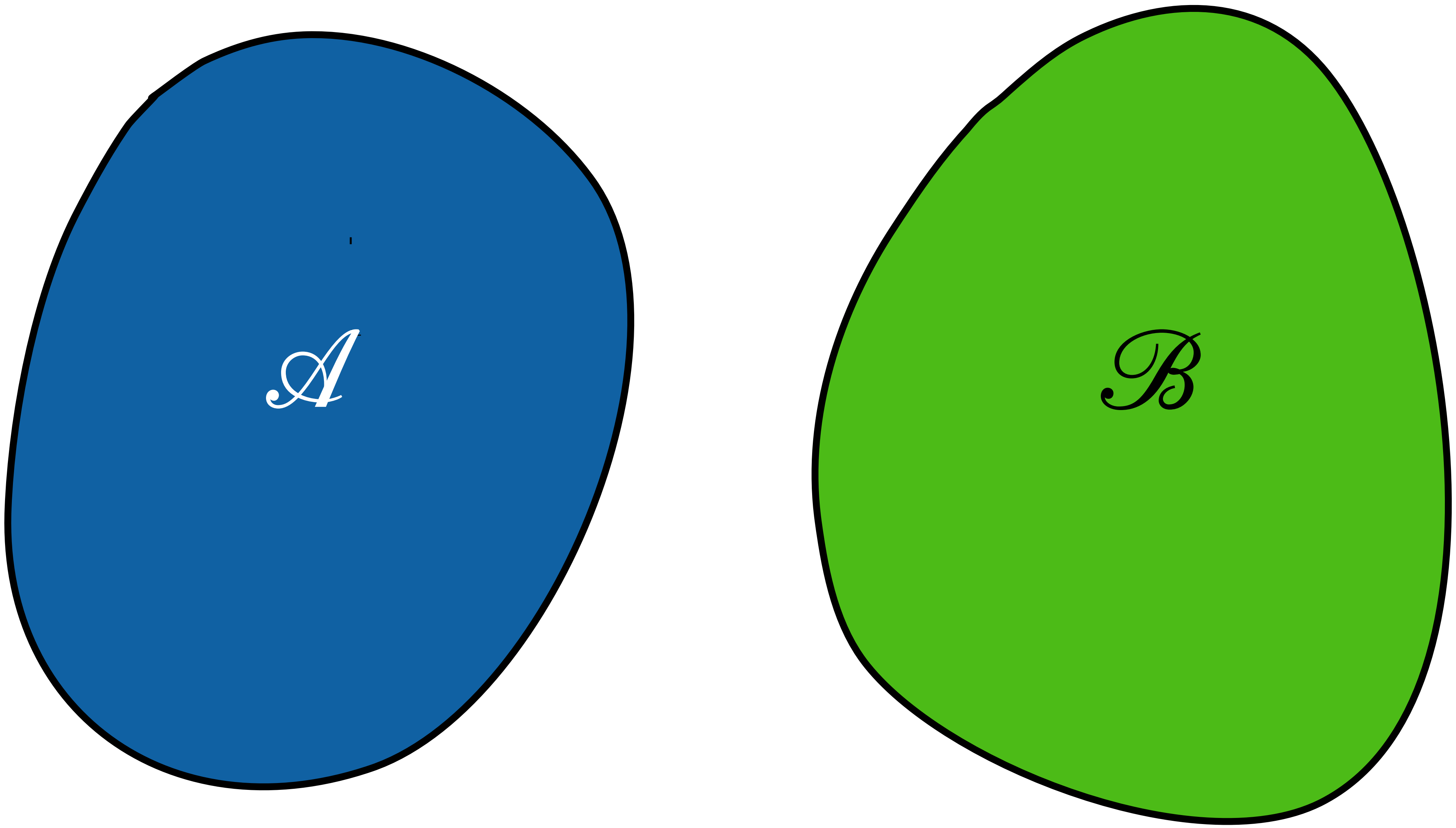


inverse relation

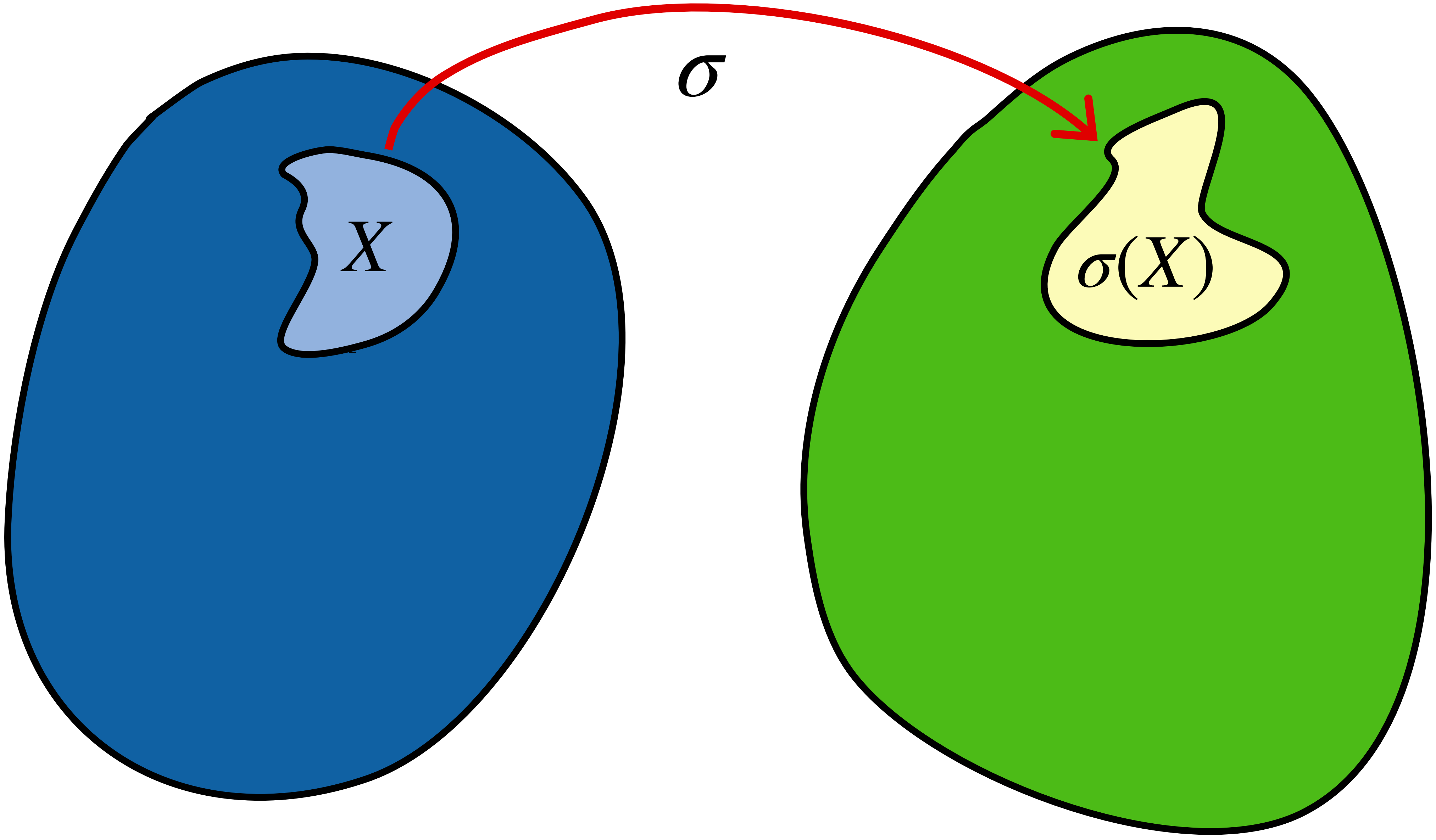


Évariste Galois
1811-1832

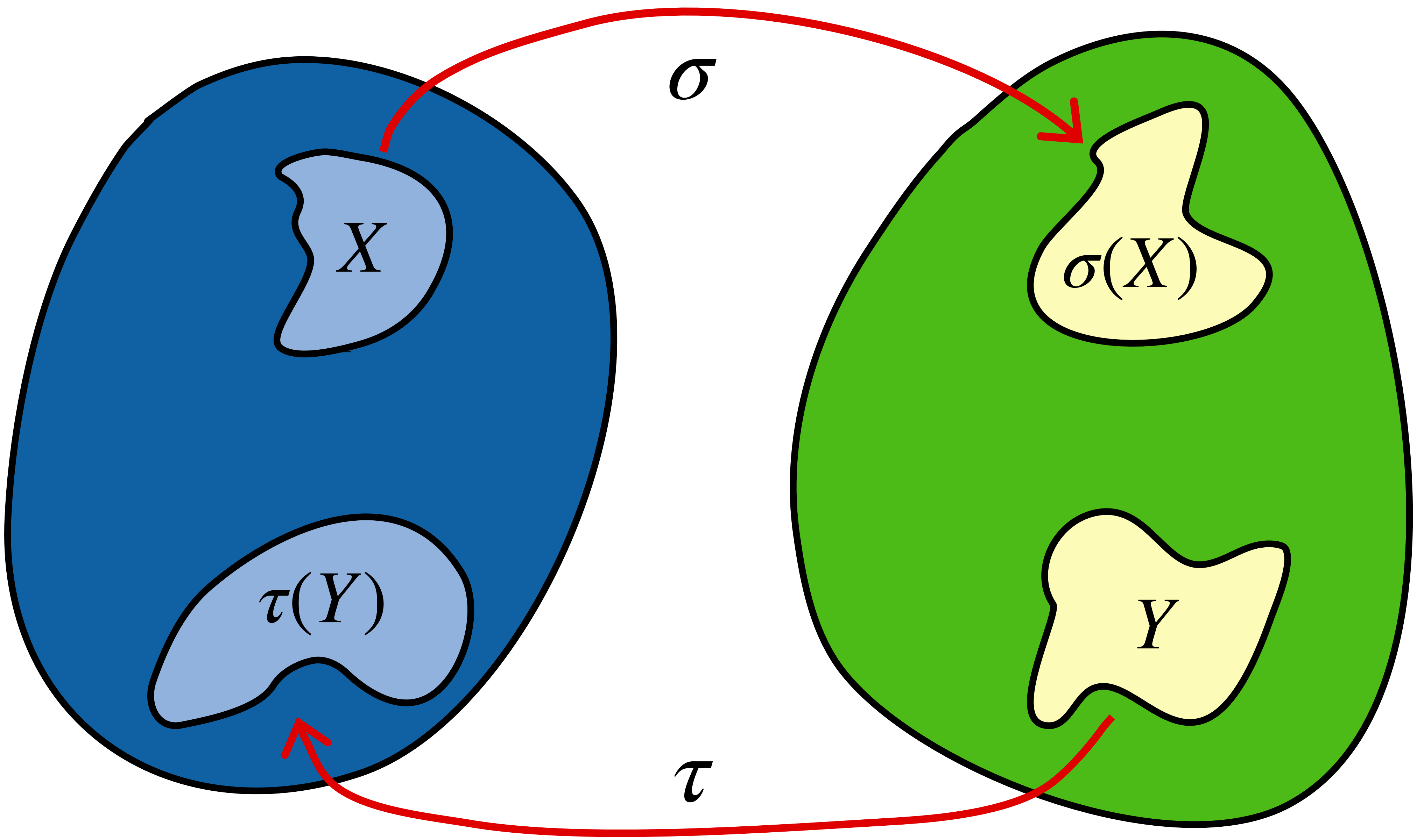
Galois connection



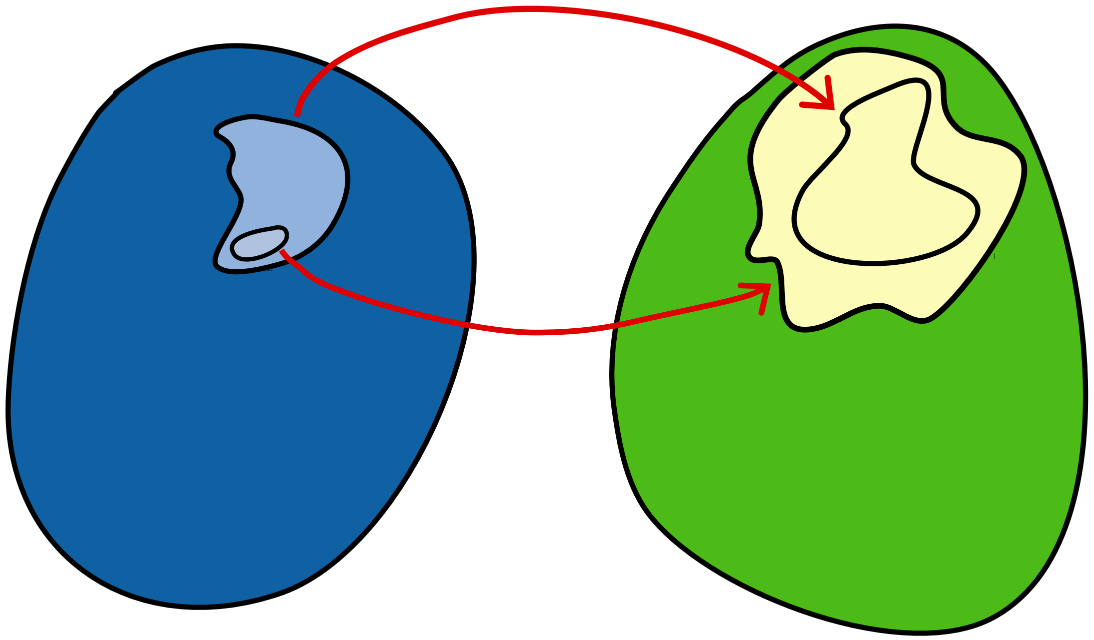
Galois connection



Galois connection



Galois connection

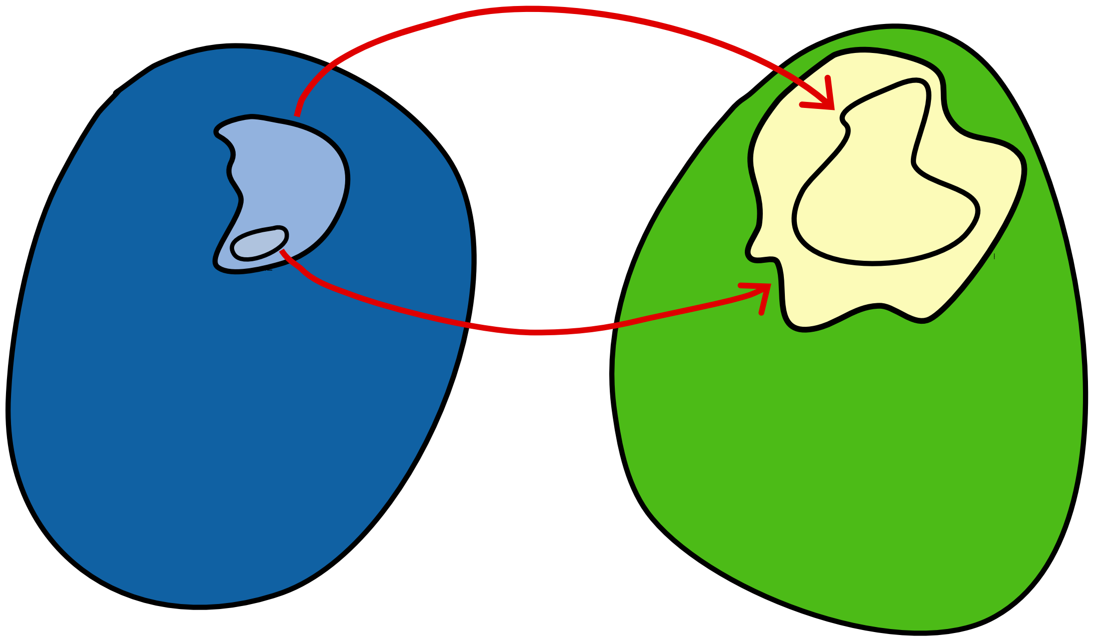


- **Condition 1**

$$X' \subseteq X \;\Rightarrow\; \sigma(X') \supseteq \sigma(X)$$

$$Y' \subseteq Y \;\Rightarrow\; \tau(Y') \supseteq \tau(Y)$$

Galois connection



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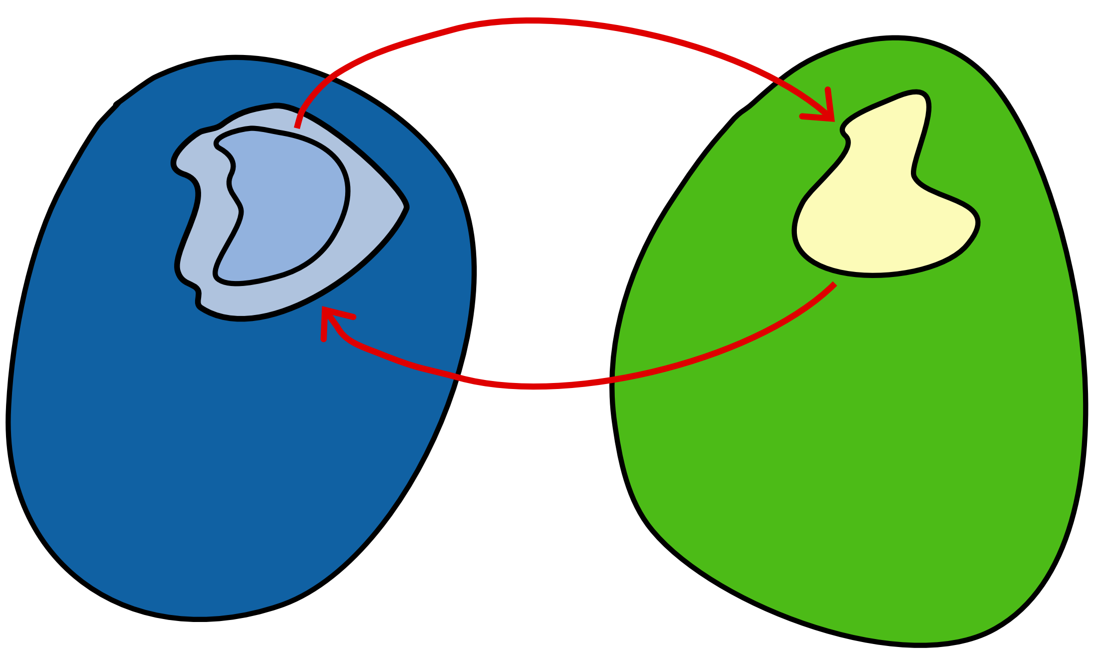


- **Condition 2**

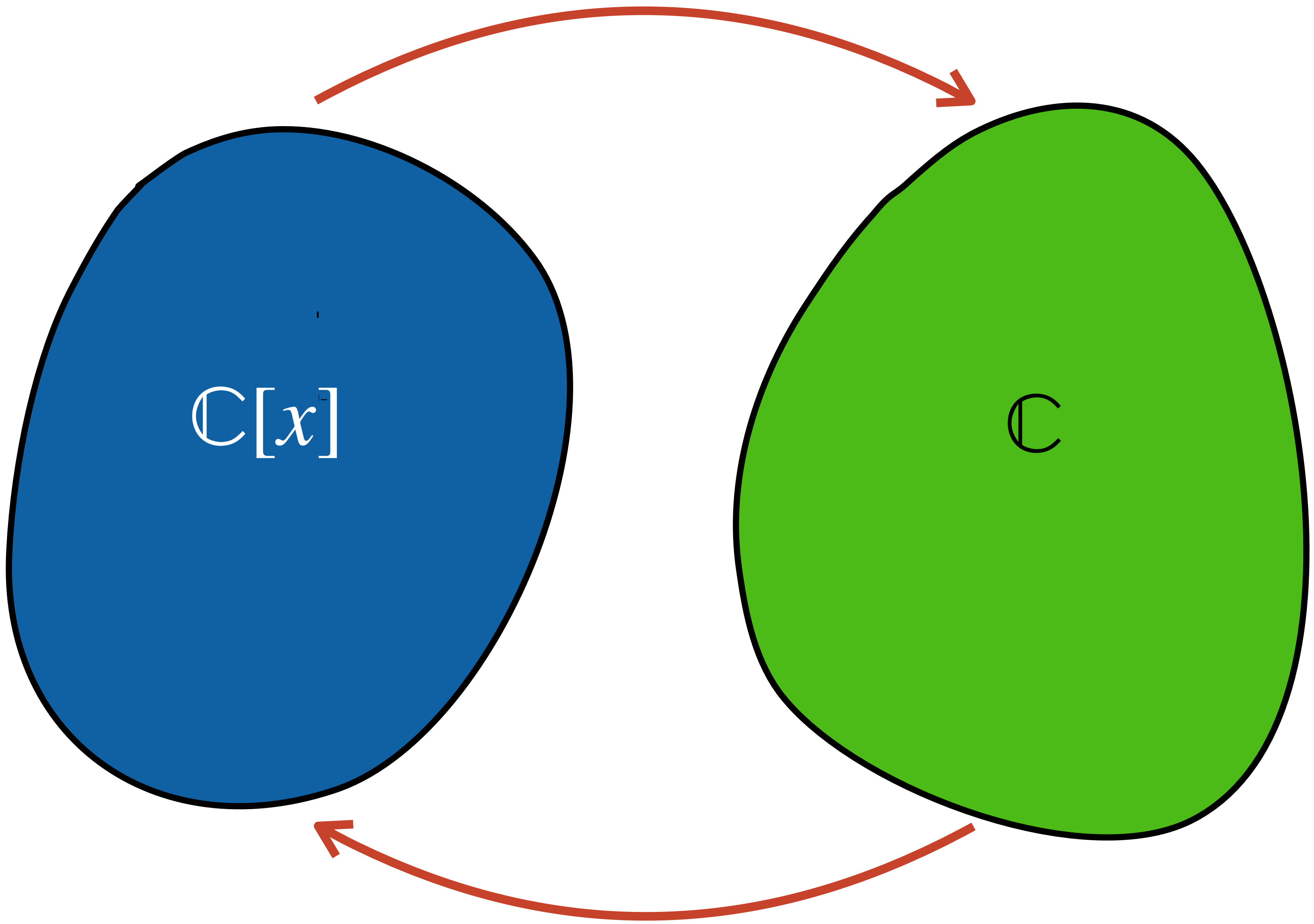
$$X \subseteq \tau(\sigma(X))$$

$$Y \subseteq \sigma(\tau(Y))$$

Galois connection

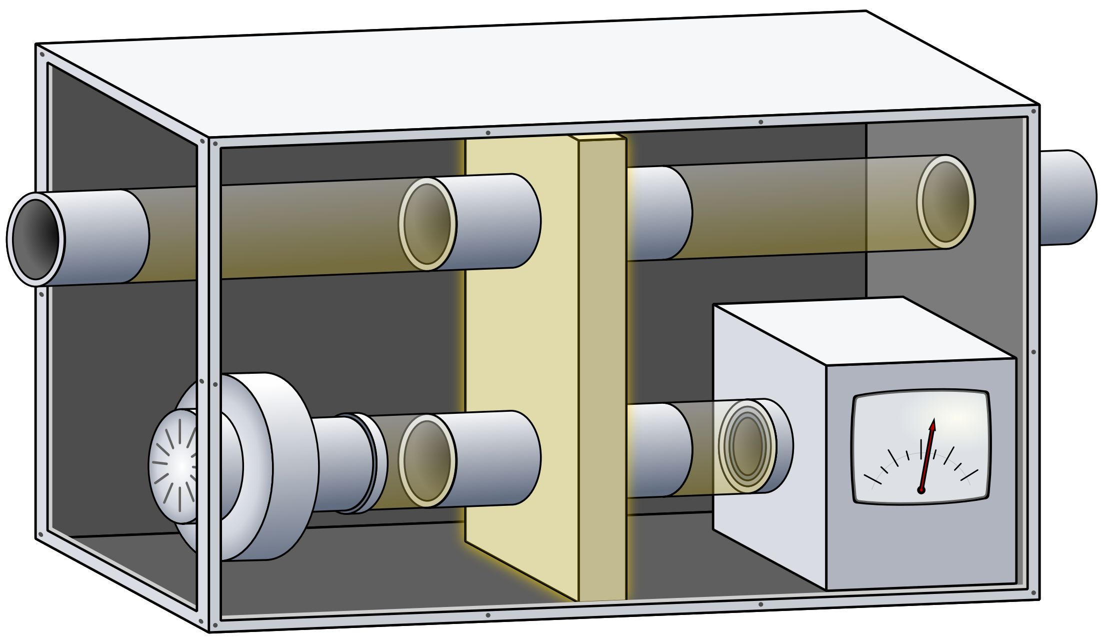
Example

$$\sigma(X) = \{c \in \mathbb{C} : f(c) = 0 \ \forall f \in X\}$$

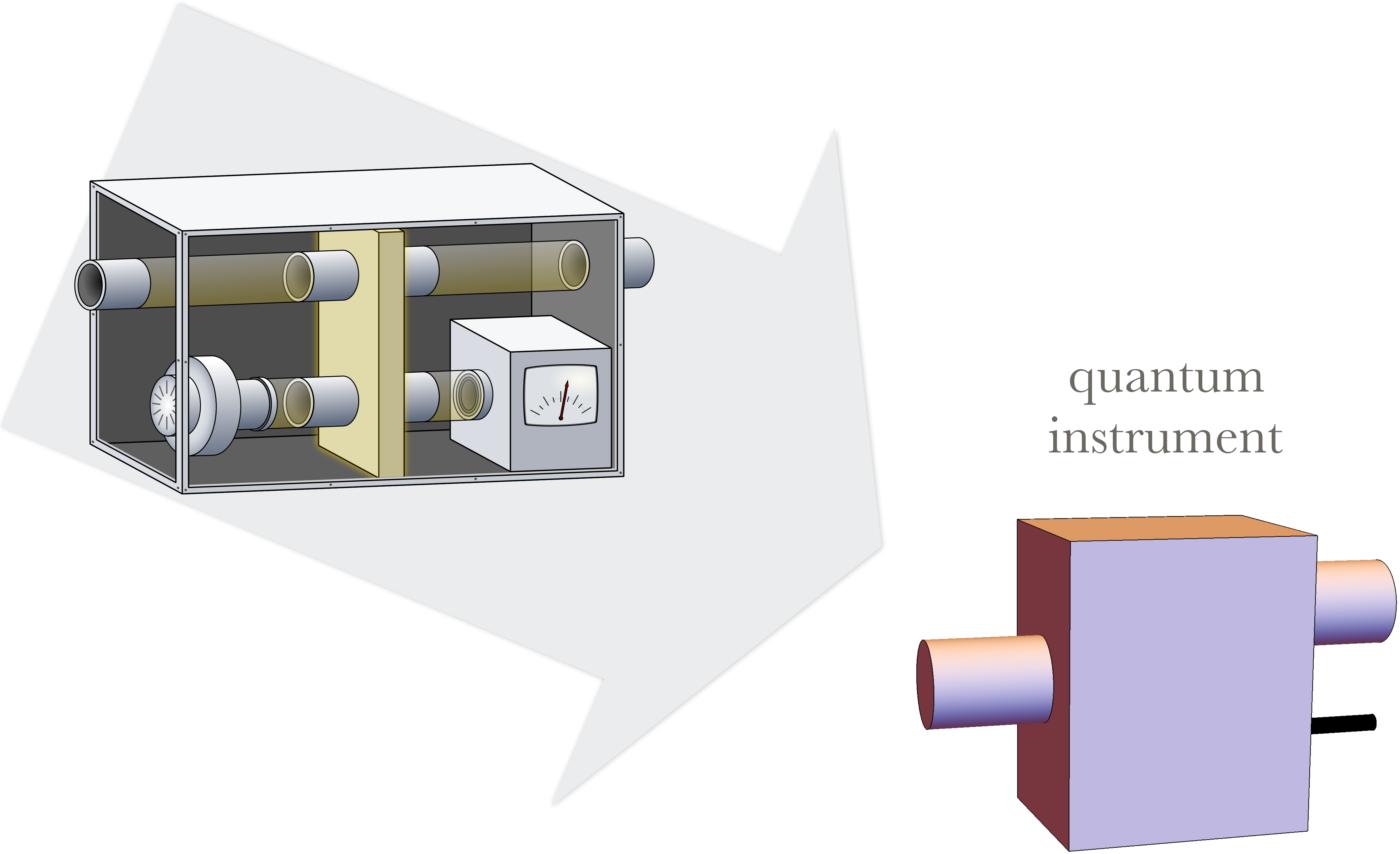


$$\tau(Y) = \{f \in \mathbb{C}[x] : f(c) = 0 \ \forall c \in Y\}$$

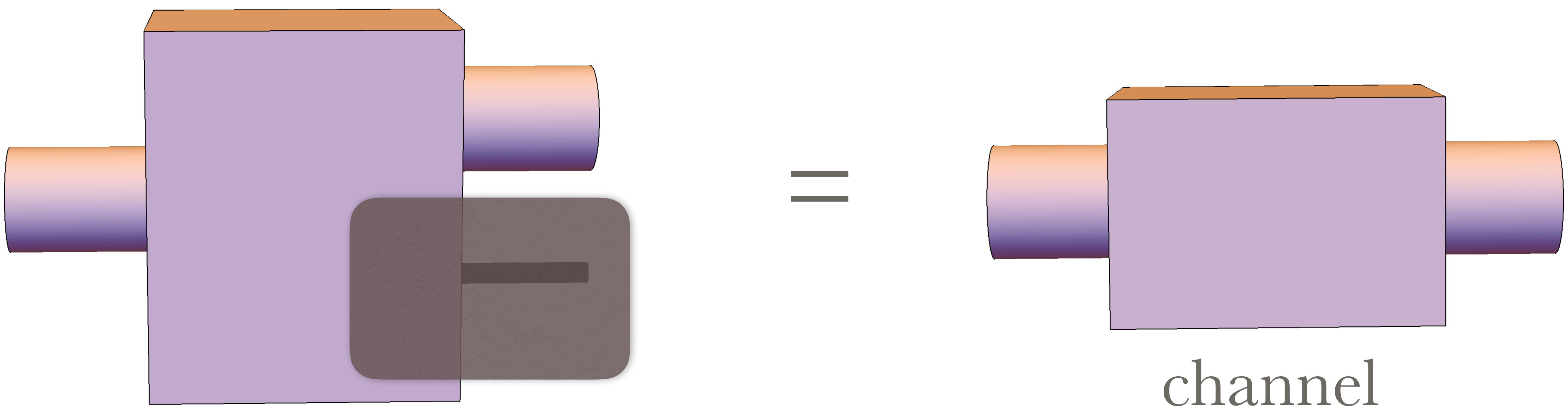
quantum measurements



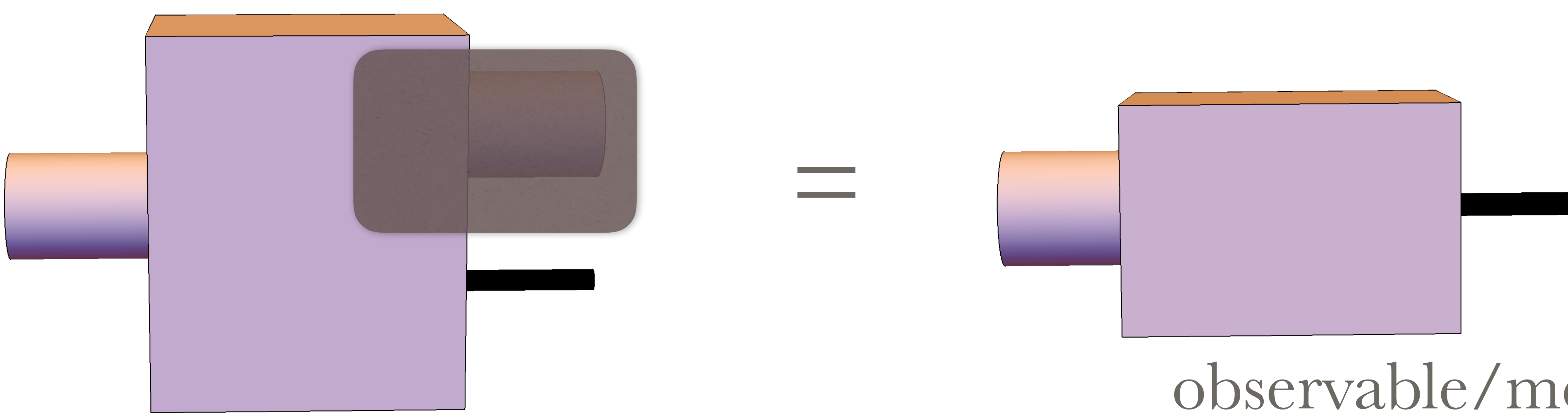
quantum measurements



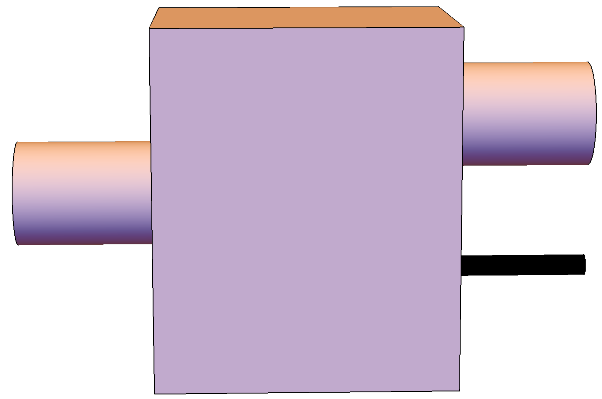
channels and observables



channel
→ *disturbance*

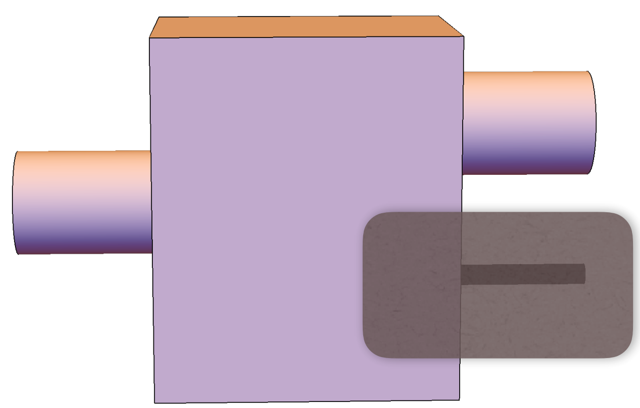


observable/meter
→ *information*

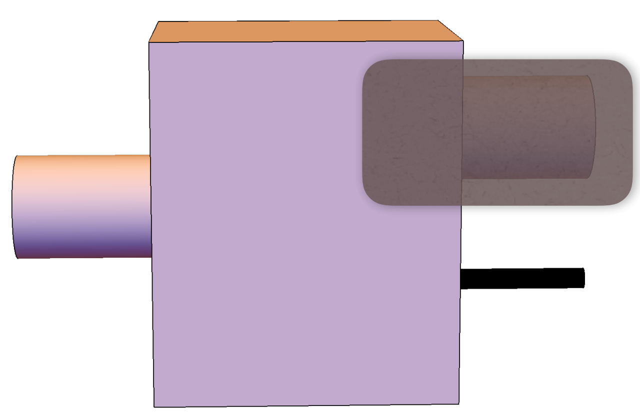


$$I(x, \varrho) = \sum_{y \in \ell_x} K_y \varrho K_y^*$$

$$K_y : \mathscr{H}_{in} \to \mathscr{H}_{out}$$

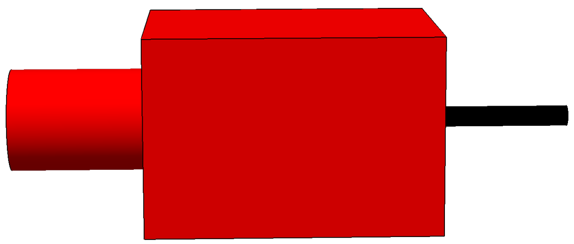
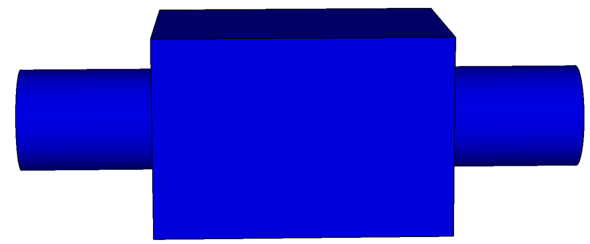


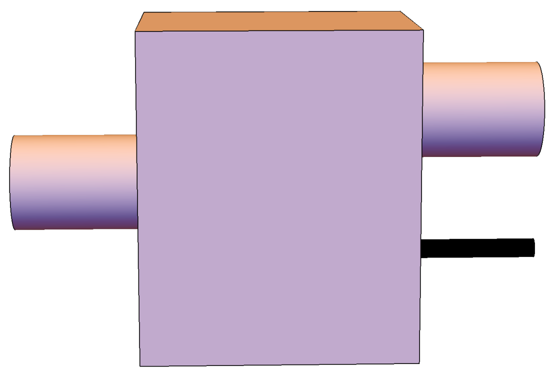
$$\Lambda(\varrho) = \sum_{y} K_y \varrho K_y^*$$ CPTP map

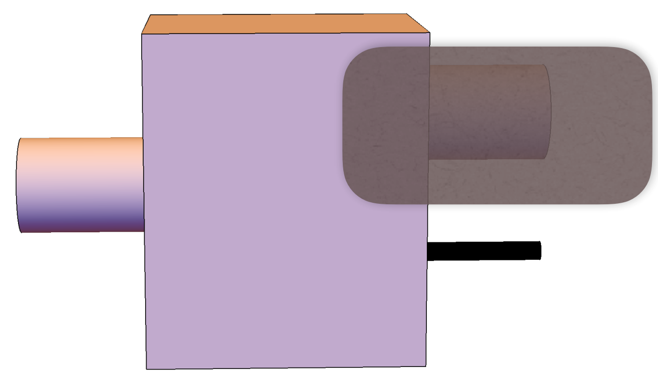
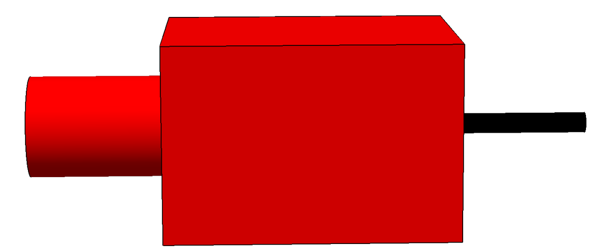


$$A(x) = \sum_{y \in \ell_x} K_y^* K_y$$ POVM

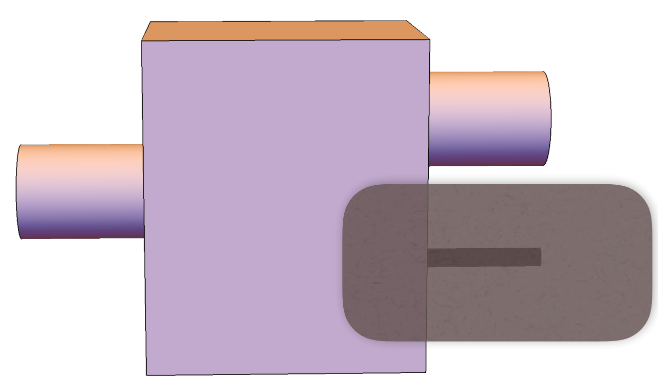
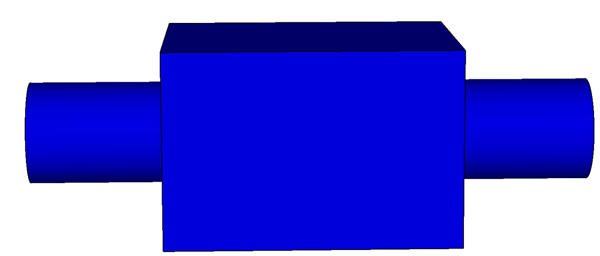
compatibility relation

 and  are **compatible**

if there exists  such that

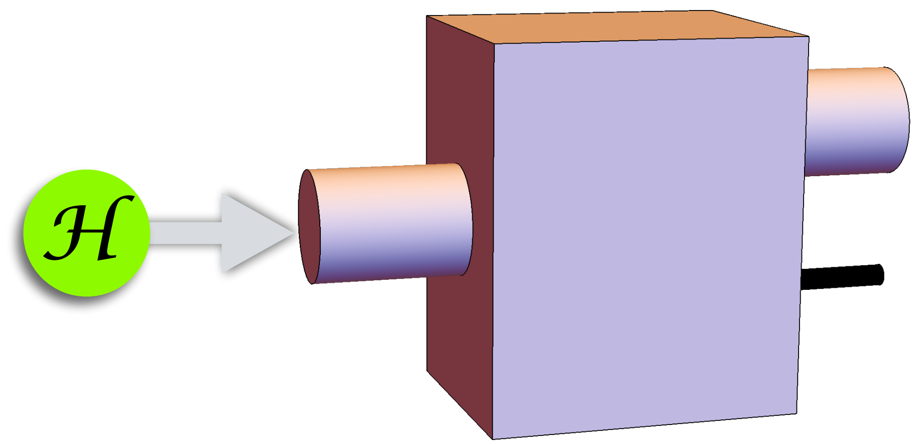
 = 

and

 = 

compatibility relation

notation

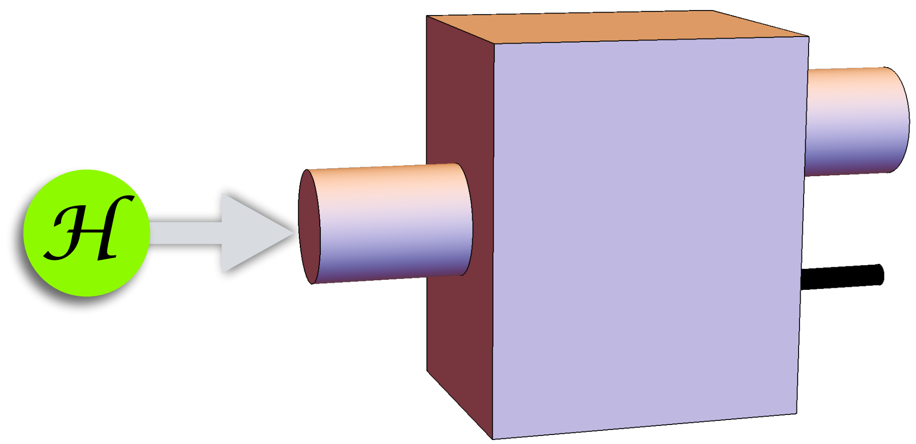


$\mathcal{O} =$ observables with input system $\mathcal{H}$, arbitrary output

$\mathcal{C} =$ channels with input system $\mathcal{H}$, arbitrary output

compatibility relation

notation



$\mathcal{O} =$ observables with input system $\mathcal{H}$, arbitrary output

$\mathcal{C} =$ channels with input system $\mathcal{H}$, arbitrary output

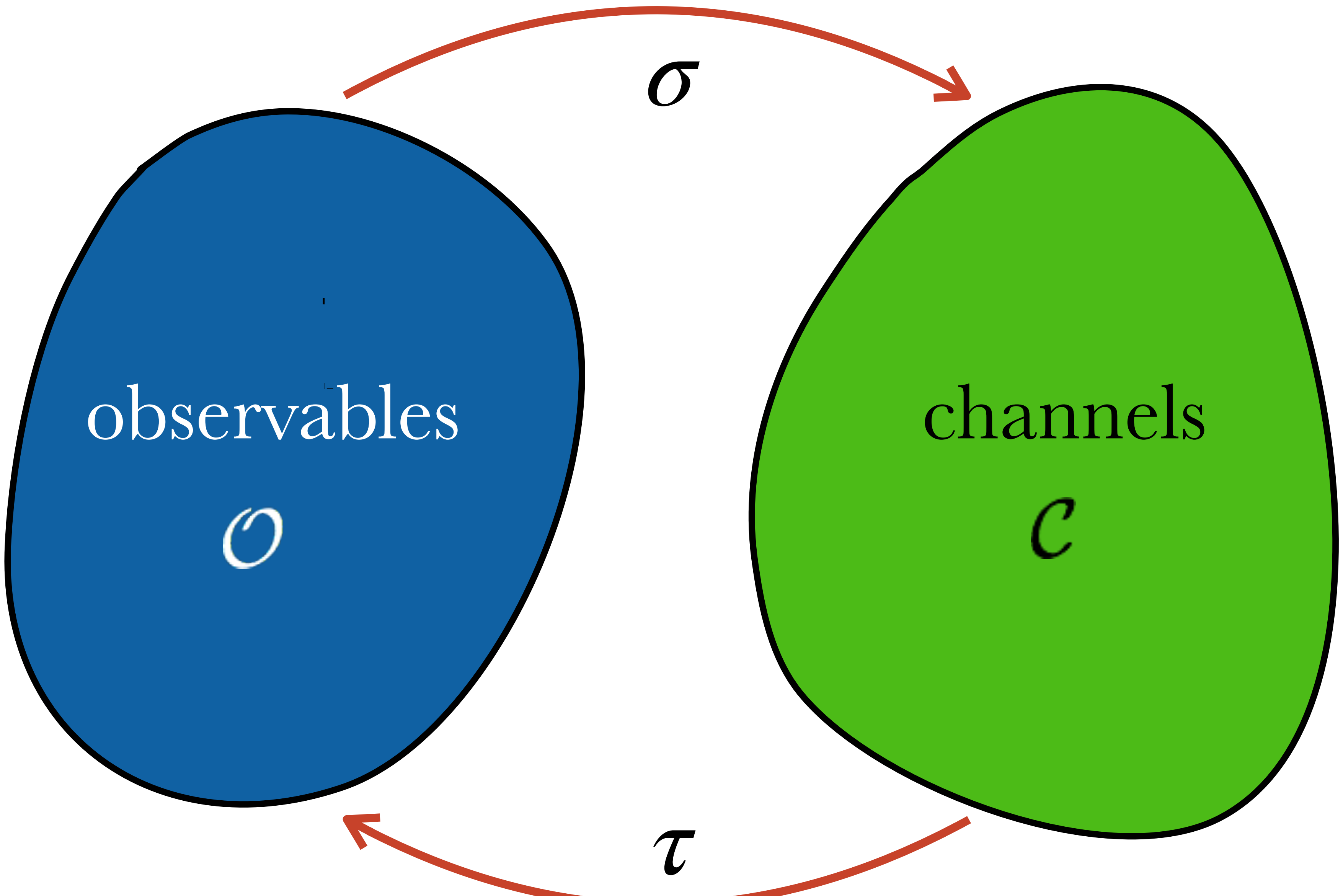
$$\sigma(A) = \{\Lambda \in \mathcal{C} \mid \Lambda \text{ is compatible with } A\}$$

$$\tau(\Lambda) = \{A \in \mathcal{O} \mid A \text{ is compatible with } \Lambda\}$$

Galois connection

$$\sigma(X) = \bigcap_{A \in X} \sigma(A) \qquad \tau(Y) = \bigcap_{\Lambda \in Y} \tau(\Lambda)$$

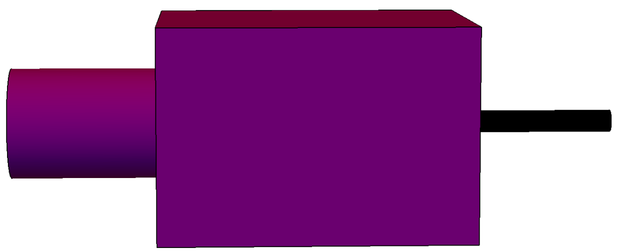
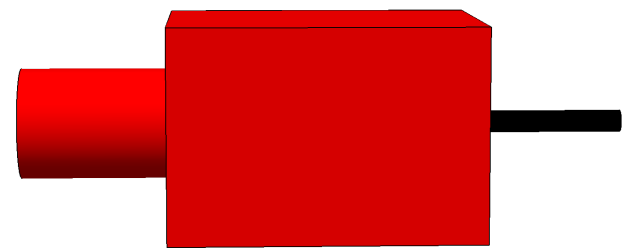
$\sigma(A) = \{\Lambda \in \mathcal{C} \mid \Lambda \text{ is compatible with } A\}$ $\tau(\Lambda) = \{A \in \mathcal{O} \mid A \text{ is compatible with } \Lambda\}$



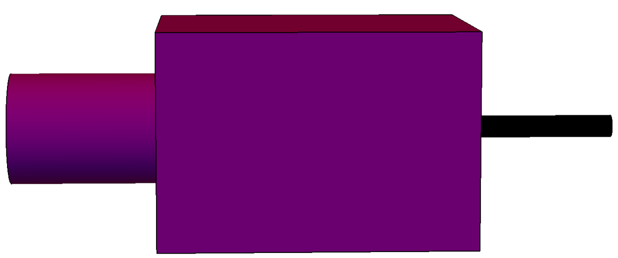
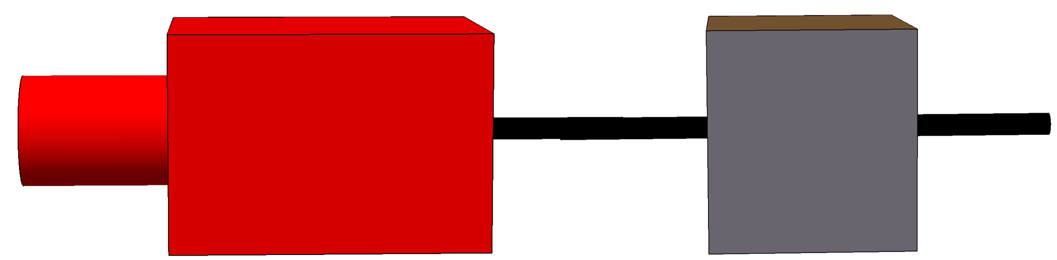
What about noise and disturbance?

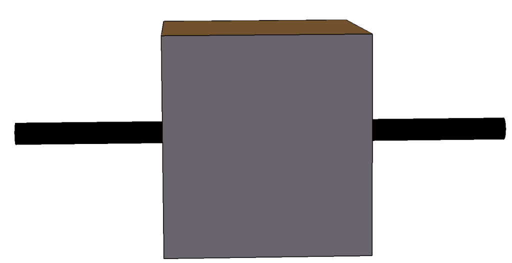
noise preorder

transitive and reflexive relation on the set of observables

 $\preccurlyeq$ 

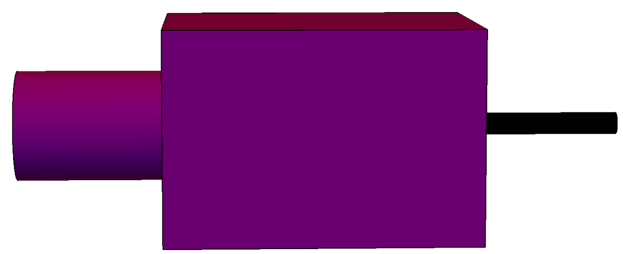
if

 $=$ 

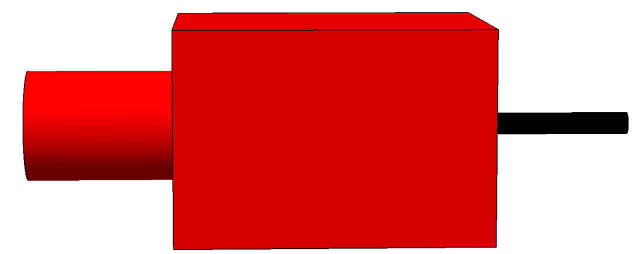
for some classical channel 

noise preorder

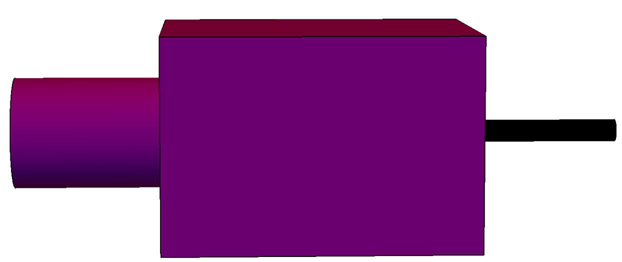
transitive and reflexive relation on the set of observables



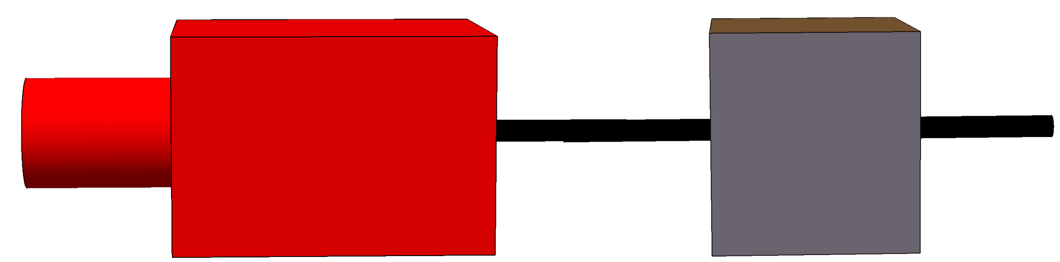
$\preccurlyeq$

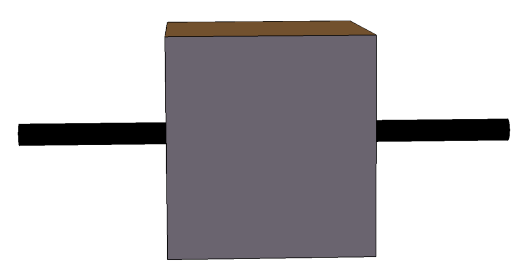


if



$=$

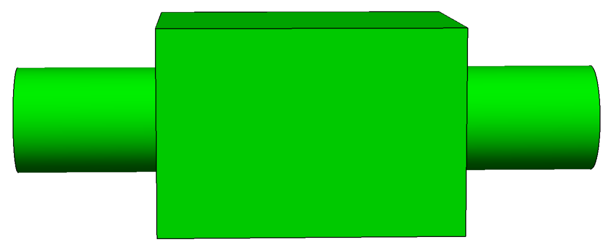
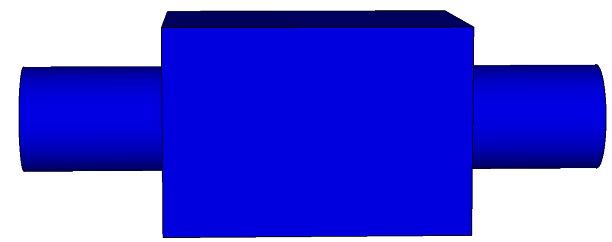


for some classical channel 

$\downarrow A = \{B \in O : B \preccurlyeq A\}$

disturbance preorder

transitive and reflexive relation on the set of channels

 $\preccurlyeq$ 

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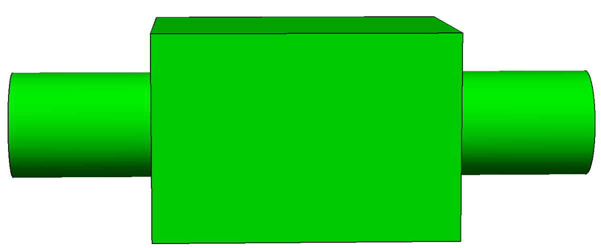
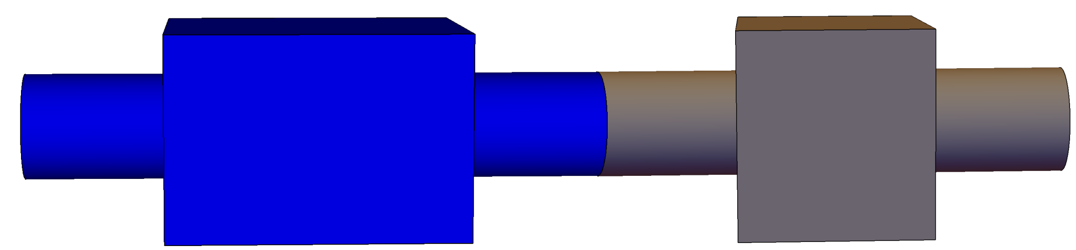
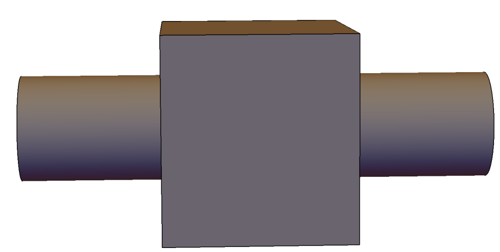
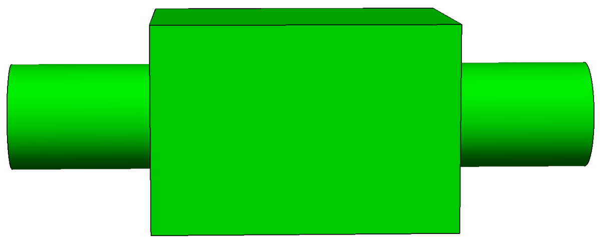
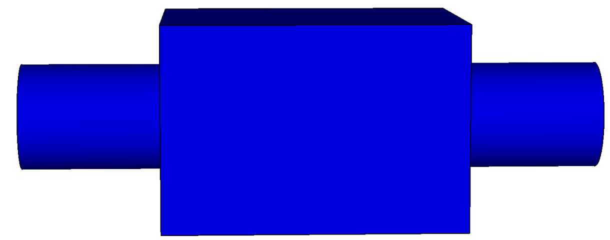
 $=$ 

for some quantum channel 

disturbance preorder

transitive and reflexive relation on the set of channels

 $\preccurlyeq$ 

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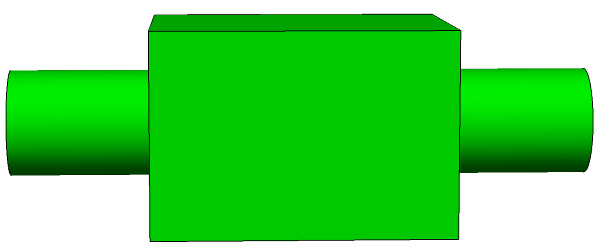
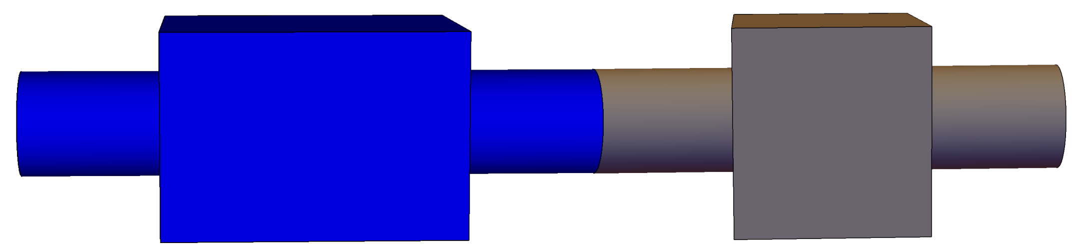
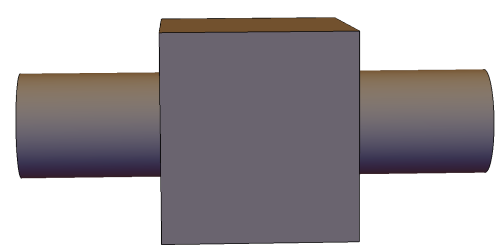
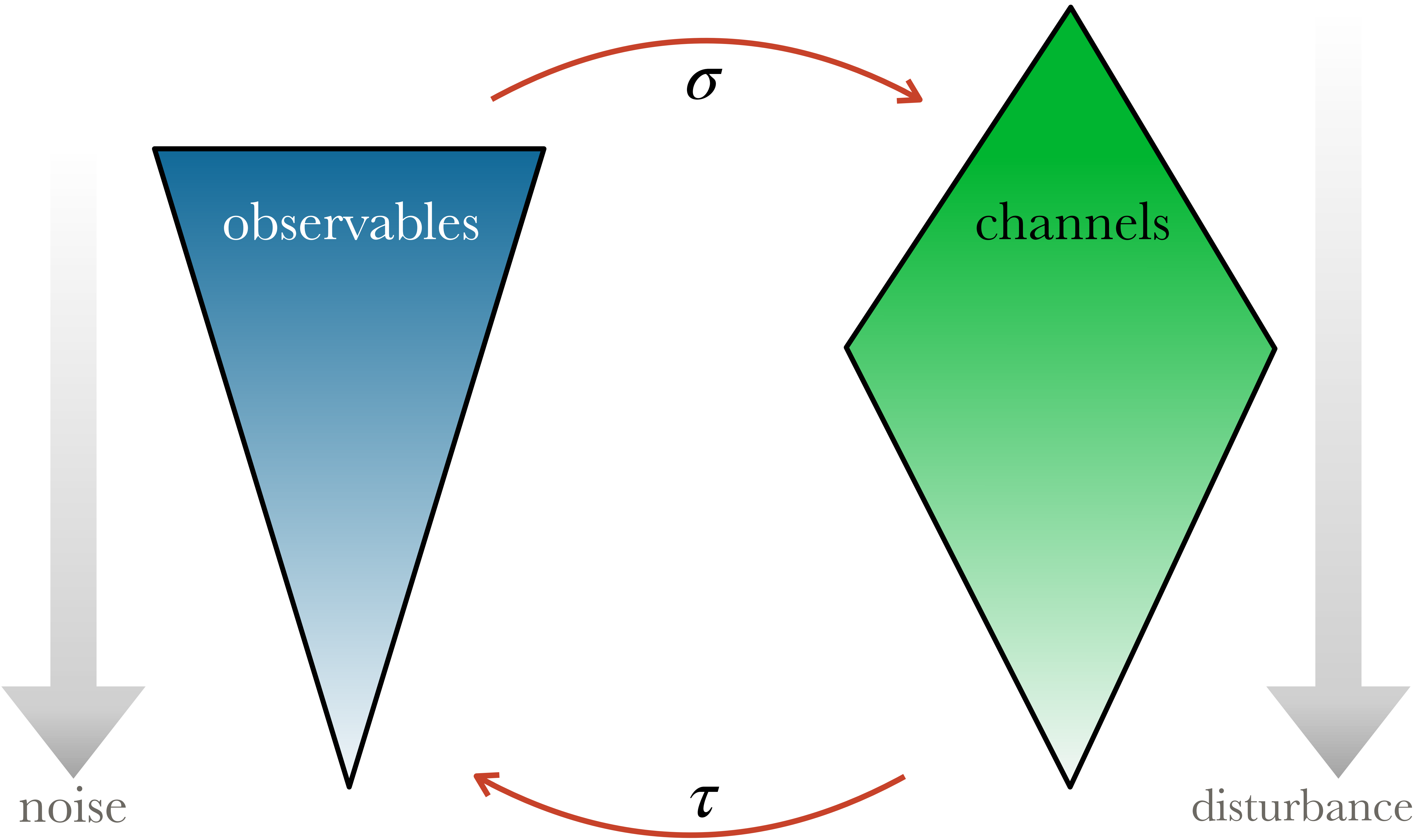
 $=$ 

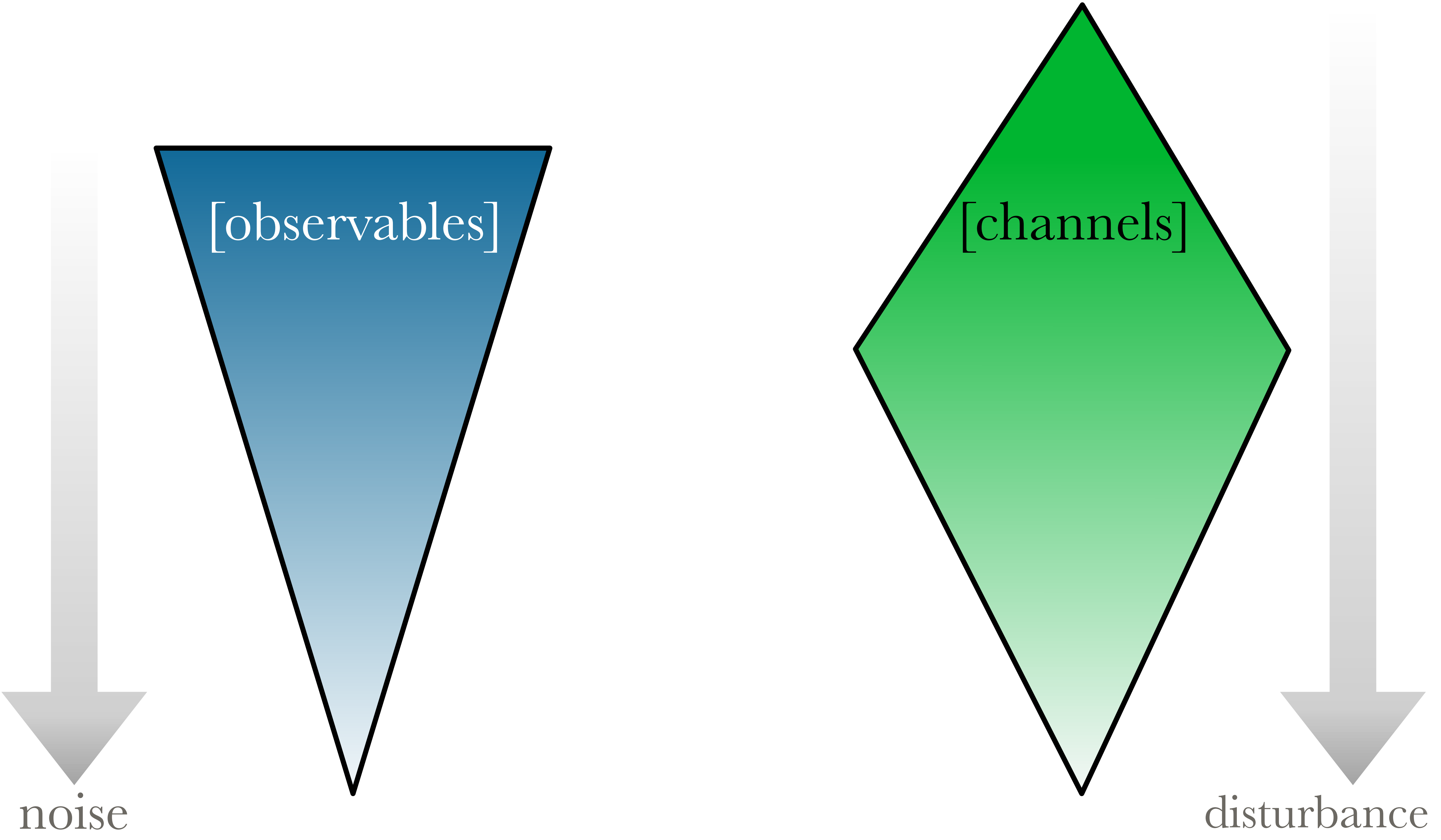
for some quantum channel 

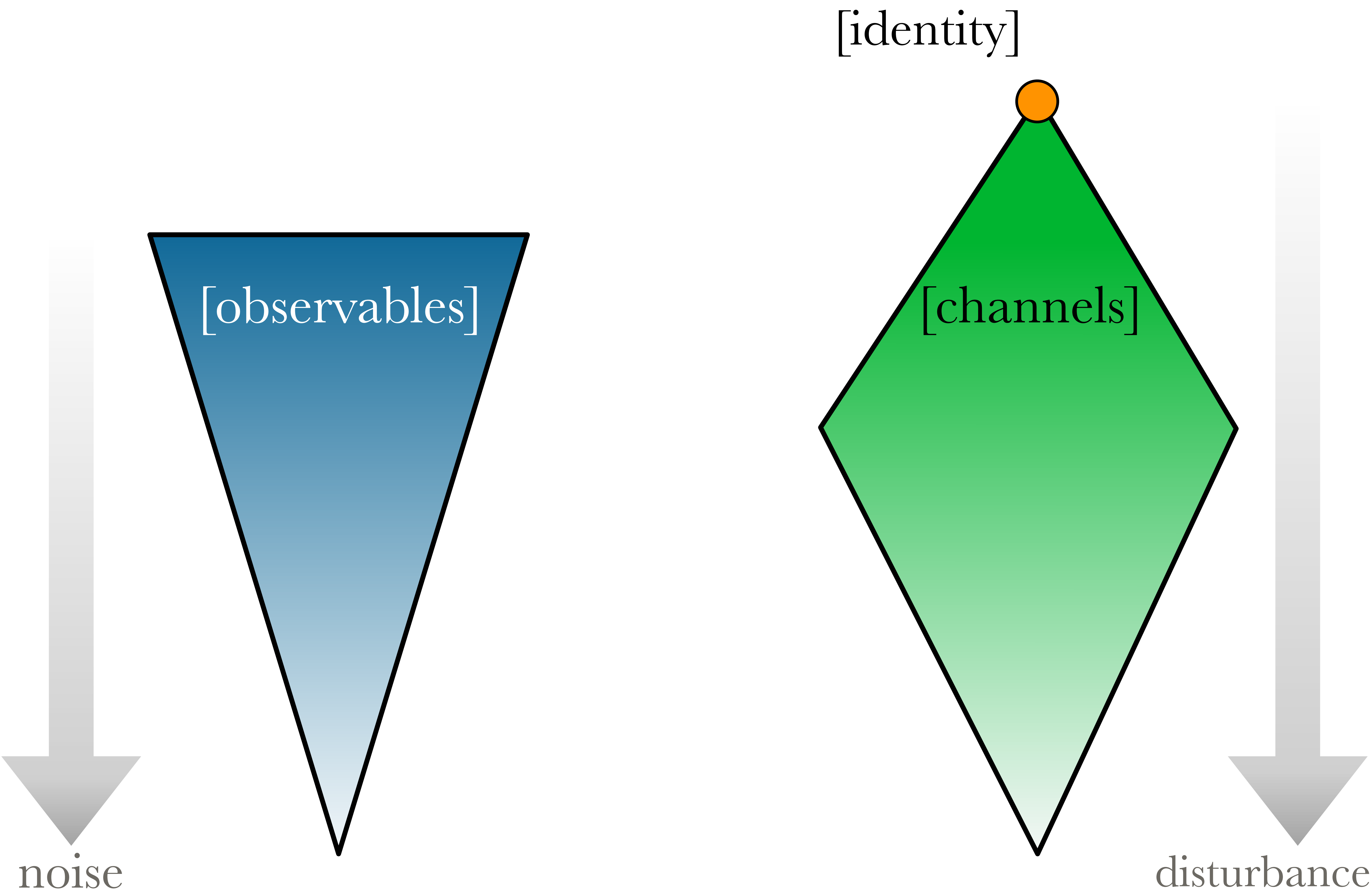
$$\downarrow \Lambda = \{\Gamma \in C : \Gamma \preccurlyeq \Lambda\}$$

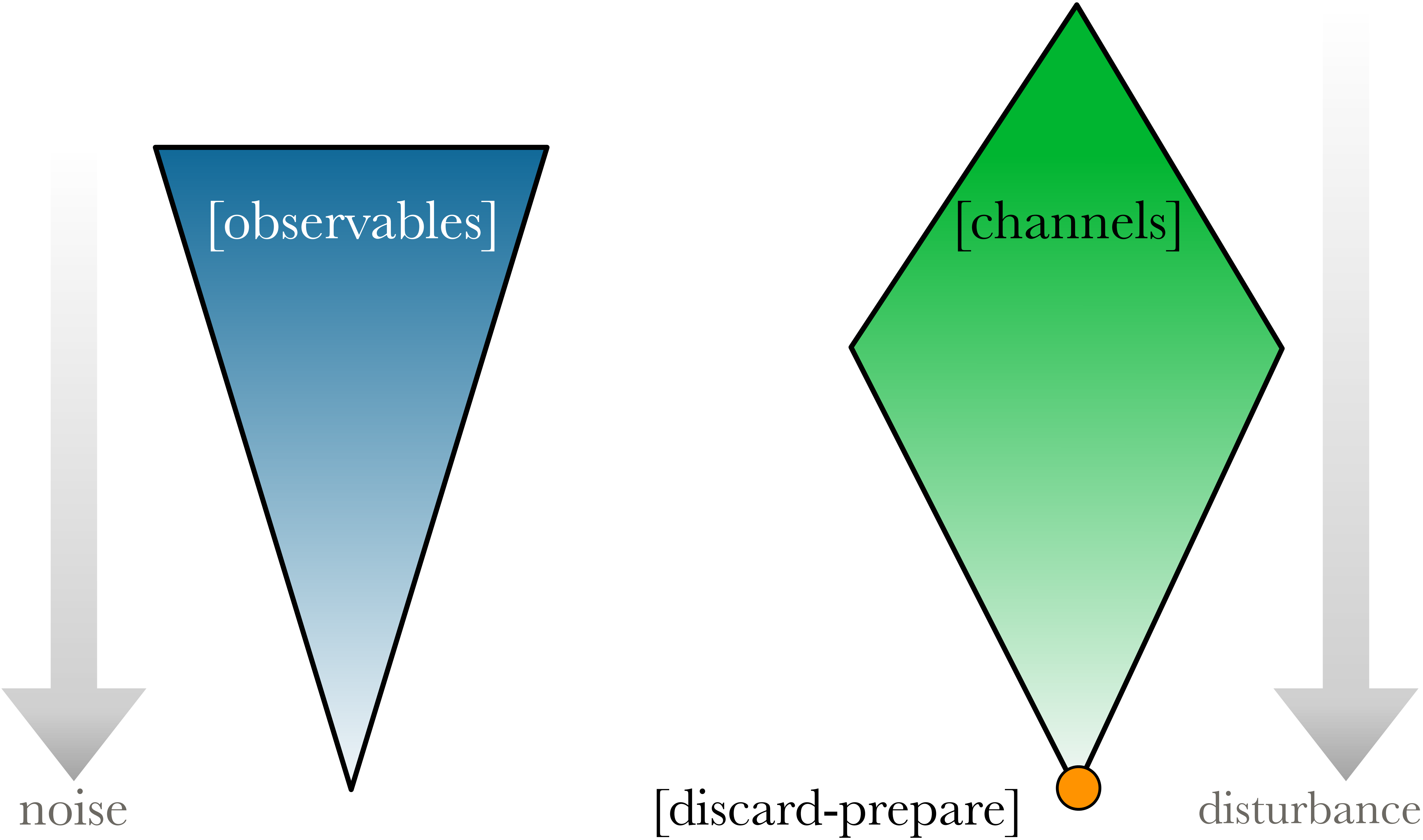
Galois connection

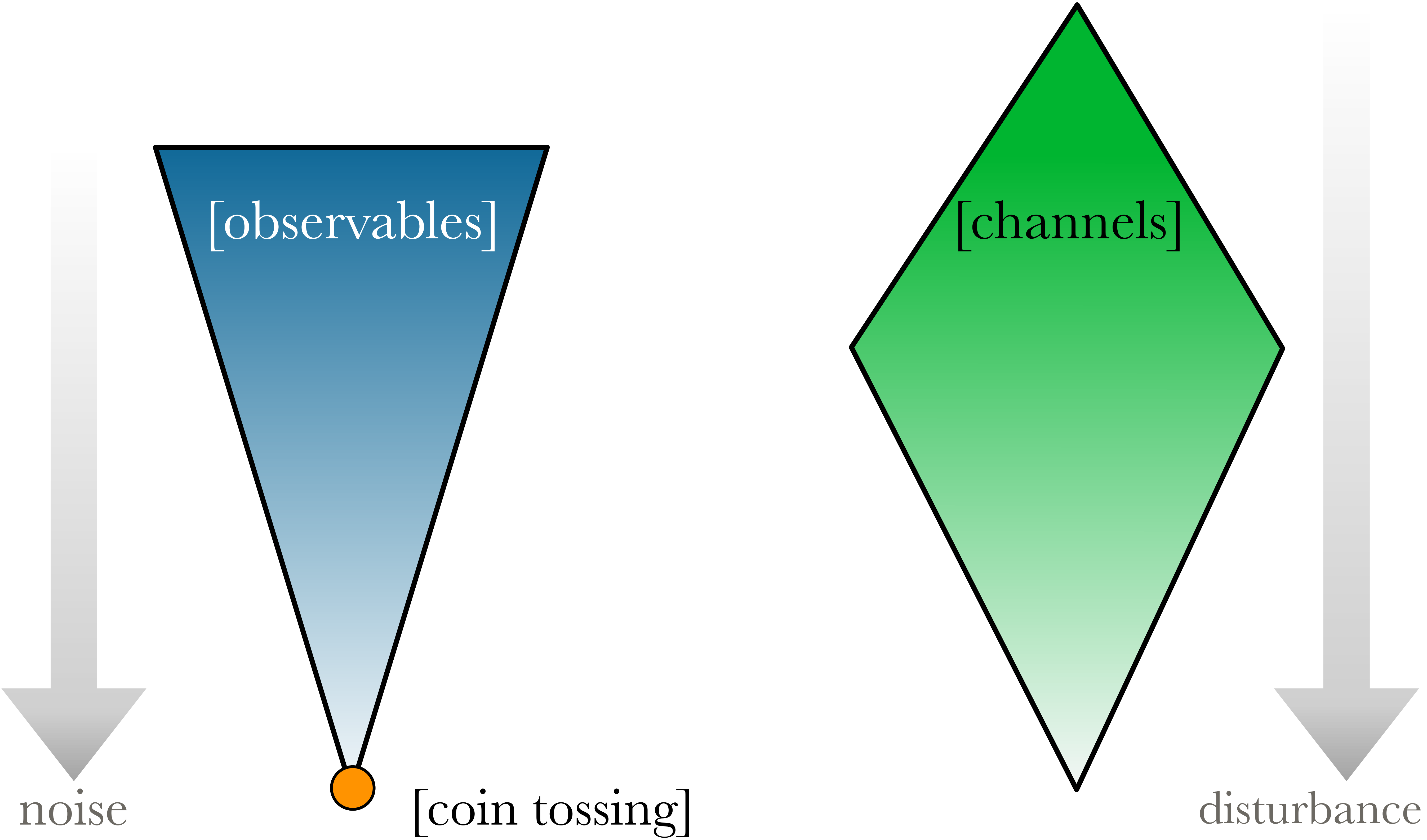


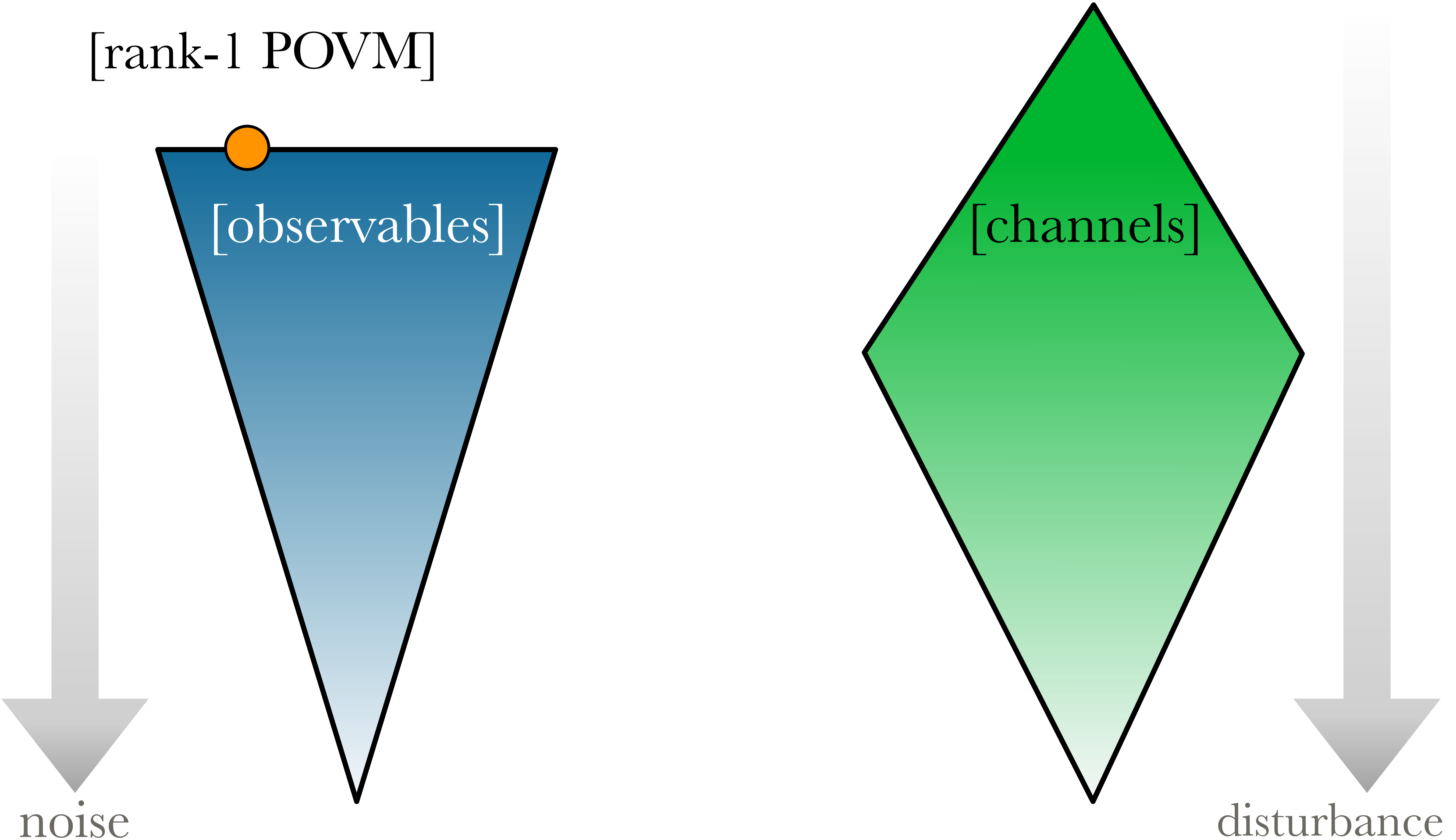
$$[A] = \{B \in O : A \preccurlyeq B \preccurlyeq A\}$$



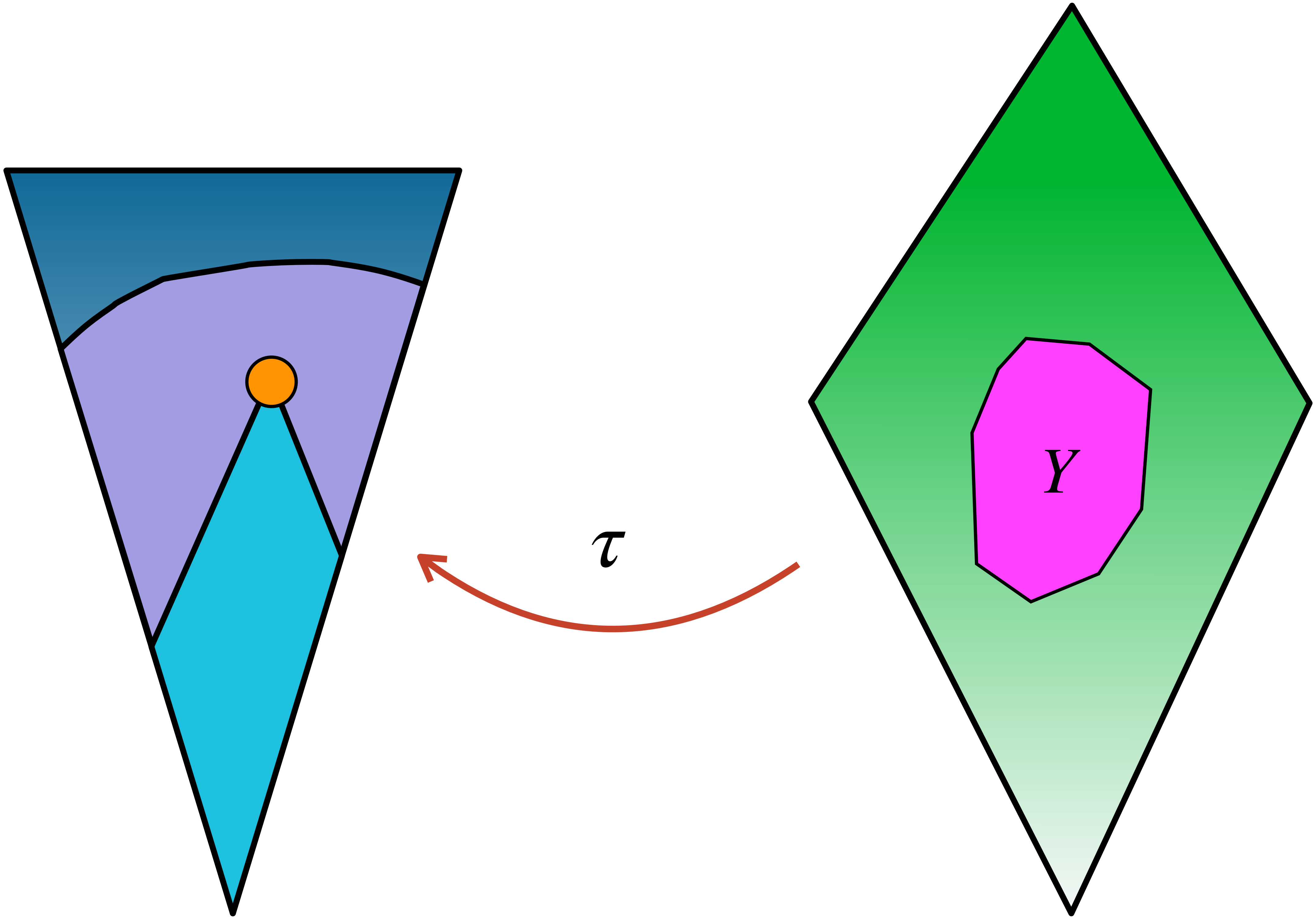

[identity]
[observables]
[channels]
noise
disturbance


[observables]
[channels]
noise
[discard-prepare]
disturbance


[observables]
[channels]
noise
[coin tossing]
disturbance


[rank-1 POVM]
[observables]
[channels]
noise
disturbance

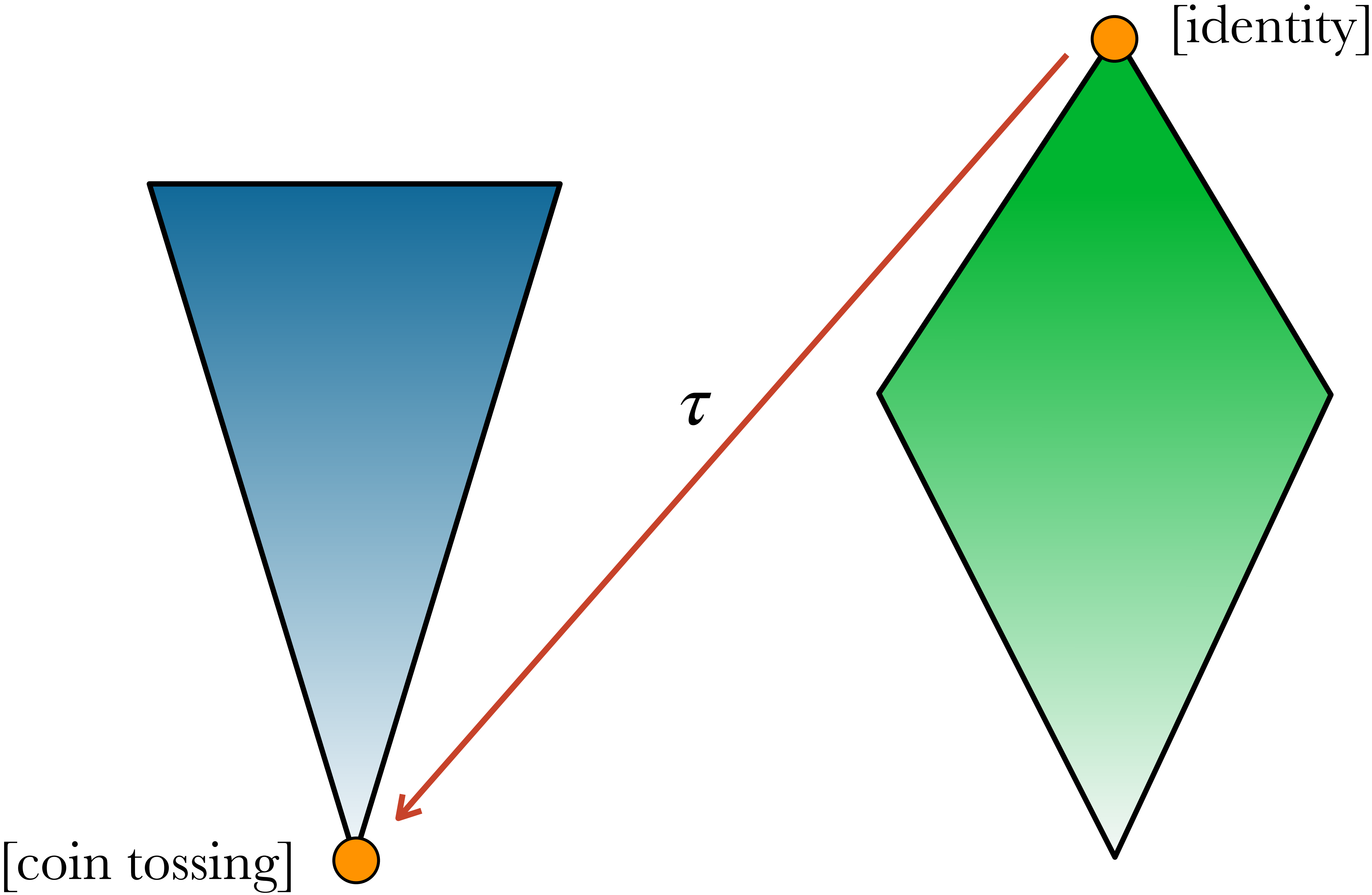
$$A \in \tau(Y) \implies {\downarrow} A \subseteq \tau(Y)$$



$$A \in \tau(Y) \quad \Longrightarrow \quad {\downarrow} A \subseteq \tau(Y)$$

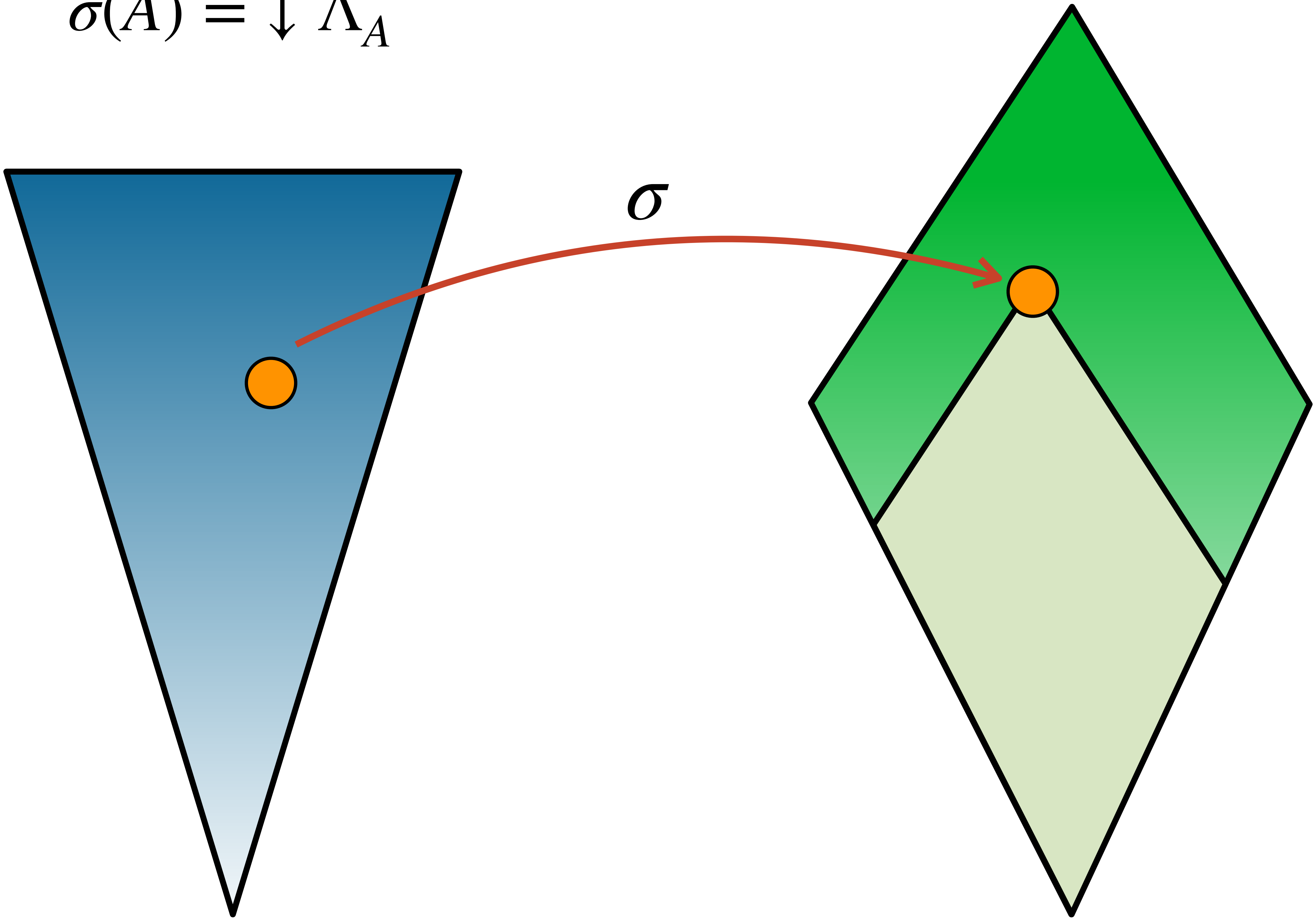
$$\Lambda \in \sigma(X) \quad \Longrightarrow \quad {\downarrow} \Lambda \subseteq \sigma(X)$$

A) no-information-without-disturbance



B) existence of a least disturbing channel

$\sigma(A) = \downarrow \Lambda_A$



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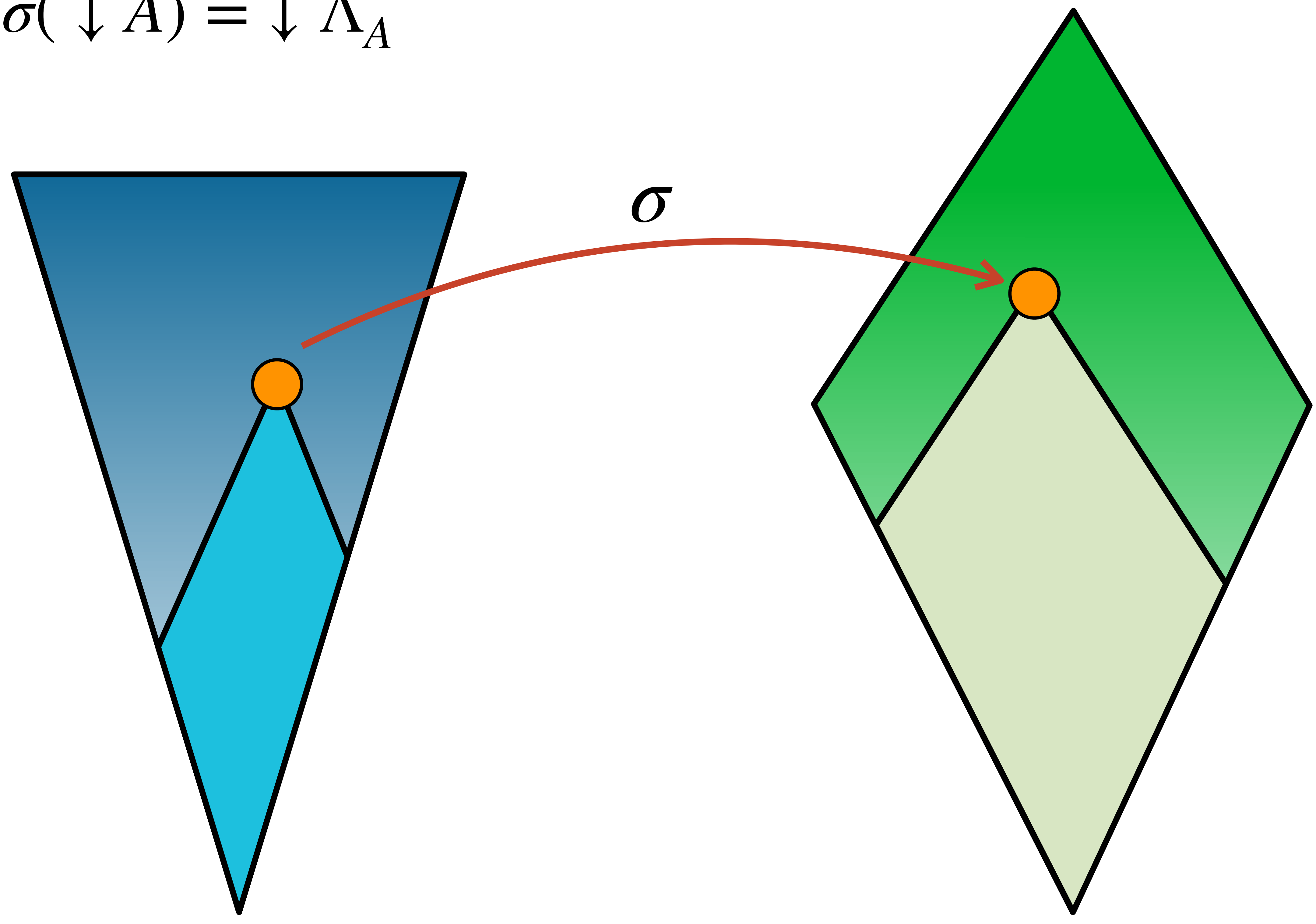
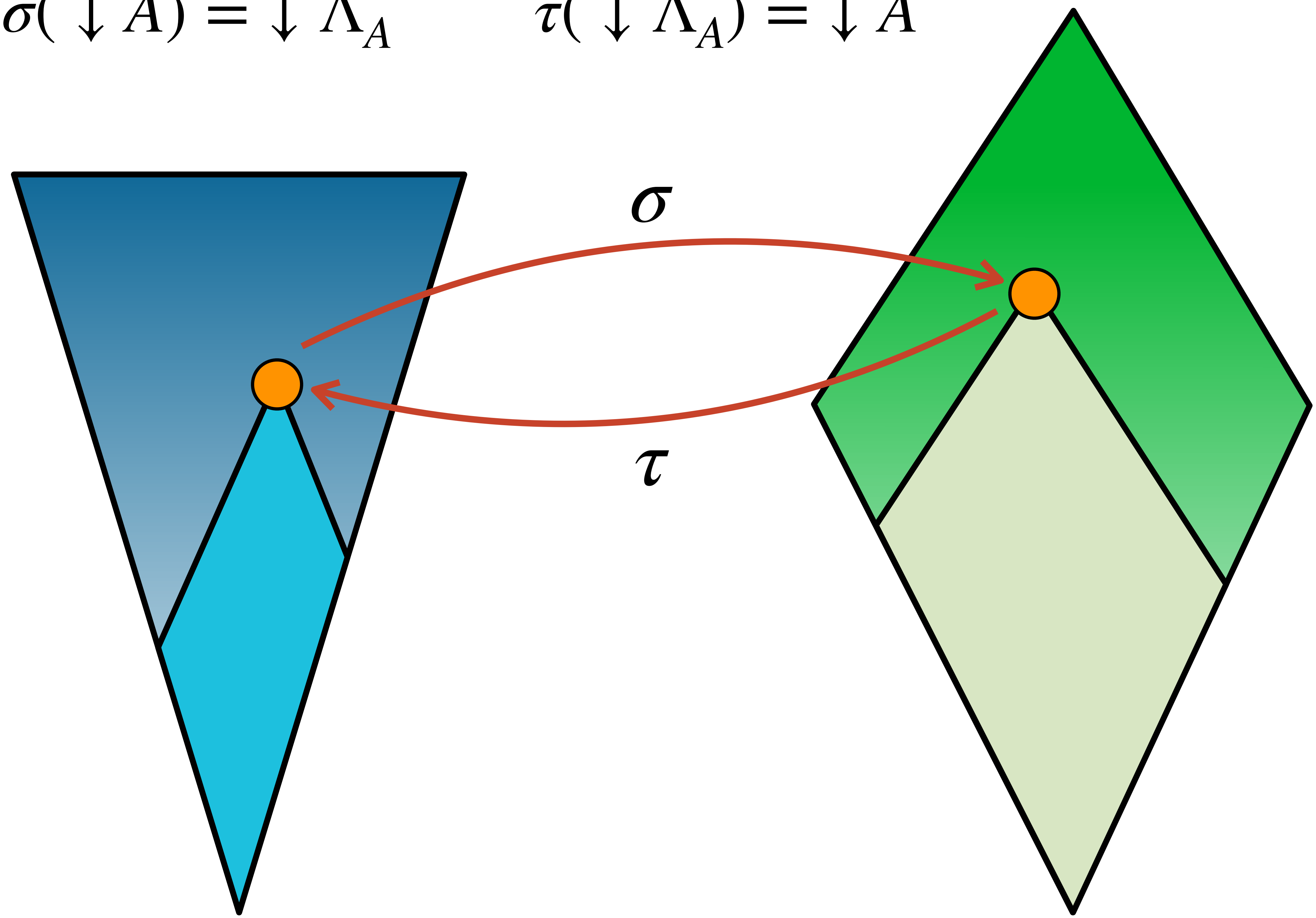
$\sigma(\downarrow A) = \downarrow \Lambda_A$



B) existence of a least disturbing channel

$\sigma(\downarrow A) = \downarrow \Lambda_A$ $\tau(\downarrow \Lambda_A) = \downarrow A$



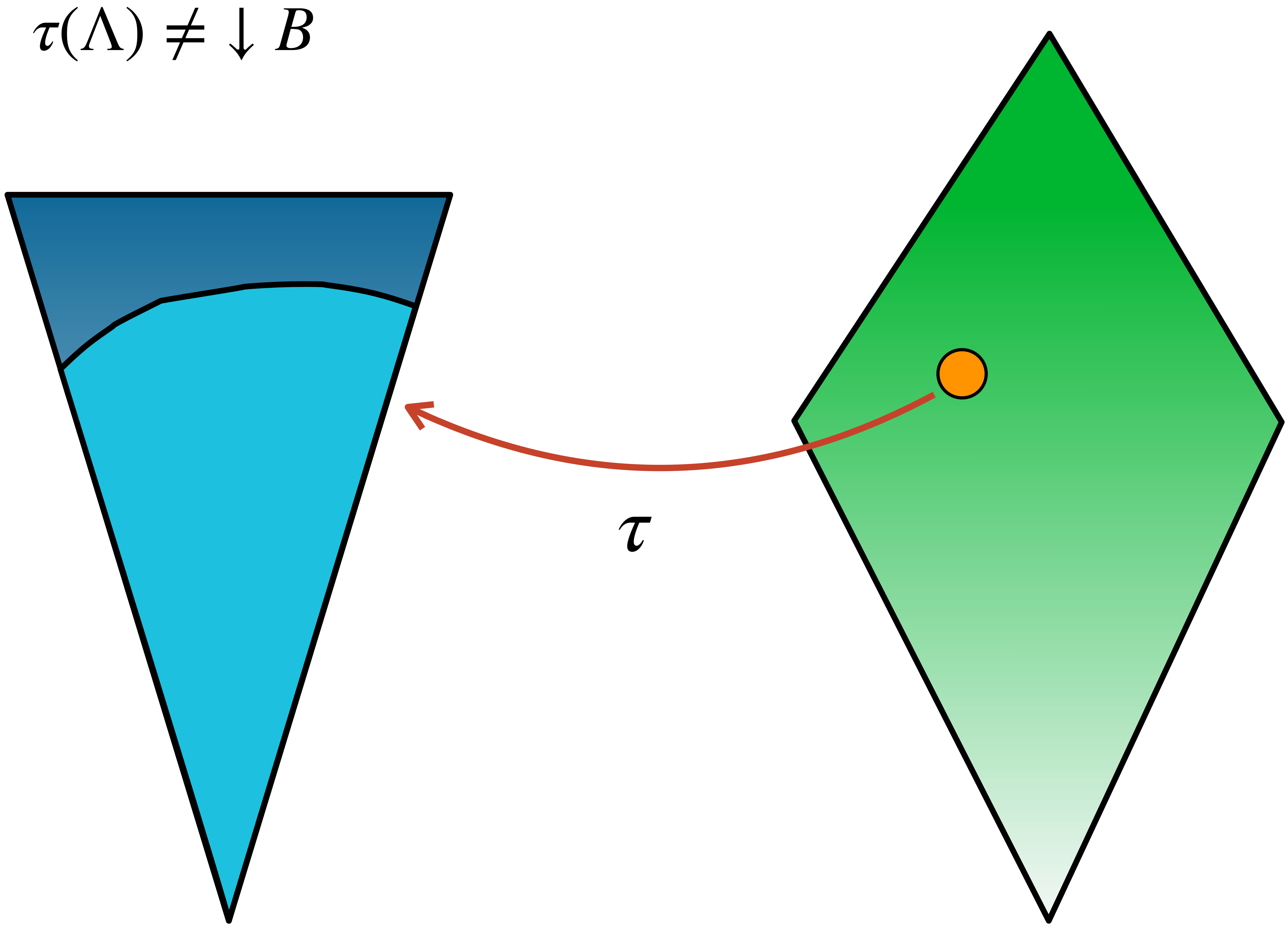
B) existence of a least disturbing channel

$$\sigma(\downarrow A) = \downarrow \Lambda_A \qquad \tau(\downarrow \Lambda_A) = \downarrow A$$

Naimark dilation: $$V^* \hat{A}(x) V = A(x)$$

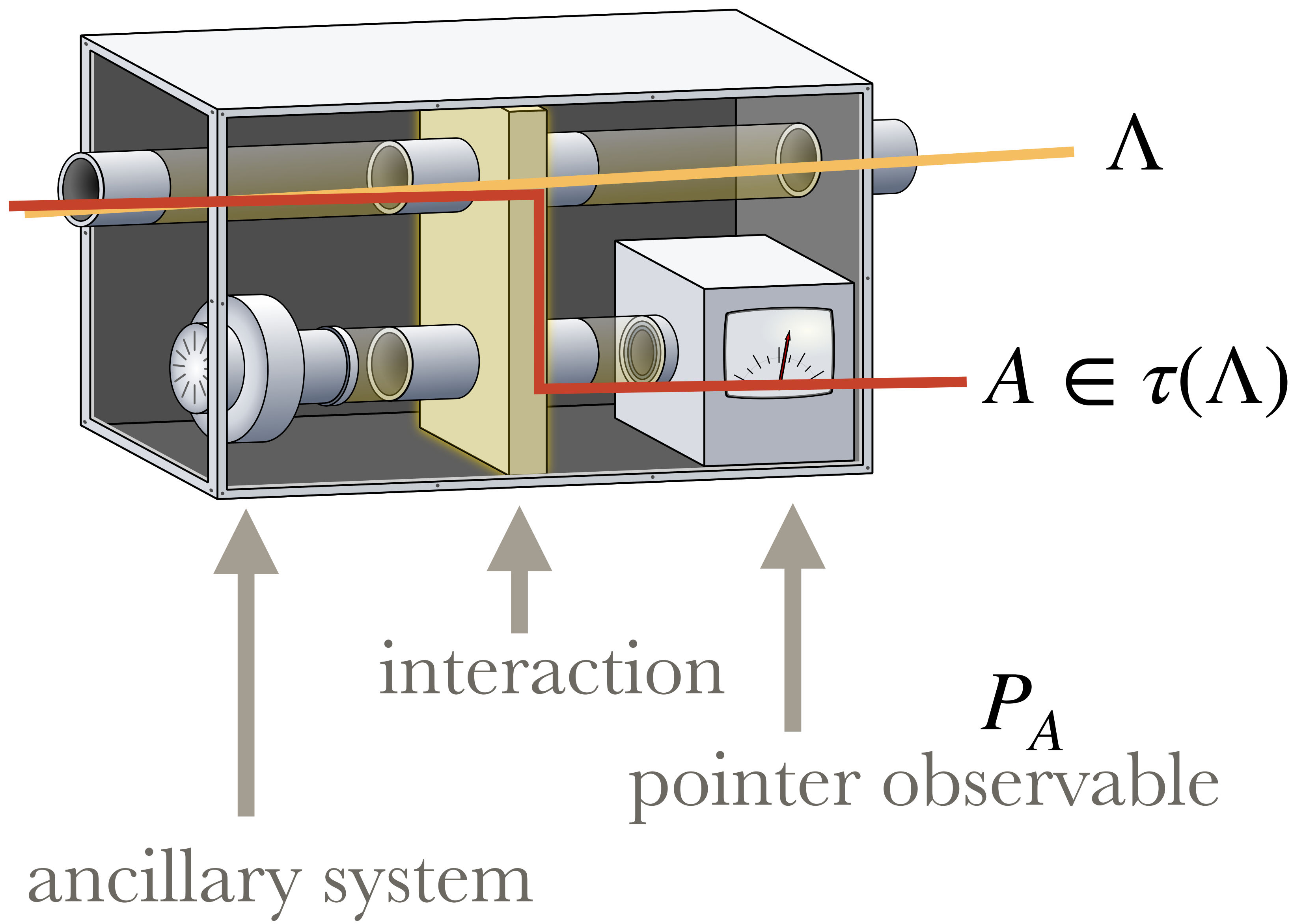
$$\Lambda_A(\varrho) = \sum_x \hat{A}(x) V \varrho V^* \hat{A}(x)$$

C) non-existence of a least noisy observable



C) non-existence of a least noisy observable

$\tau(\Lambda)$: fix a realization for Λ and go through all pointer observables



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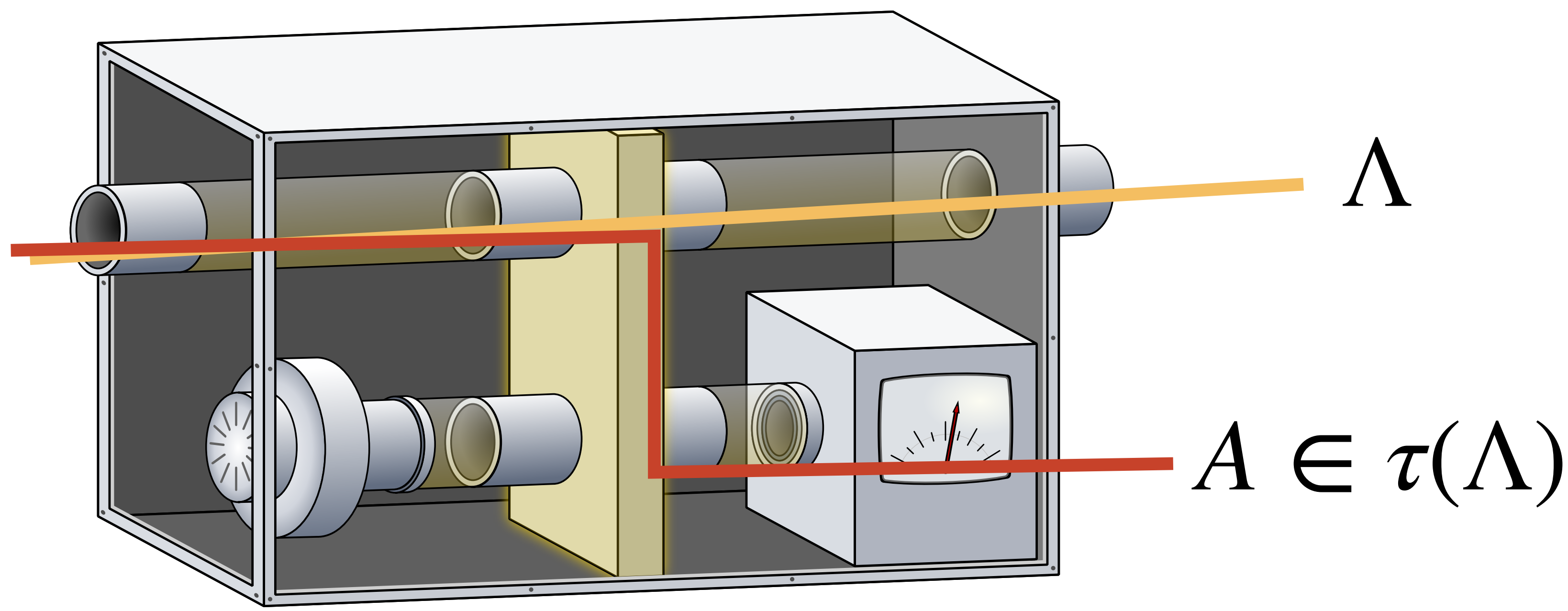
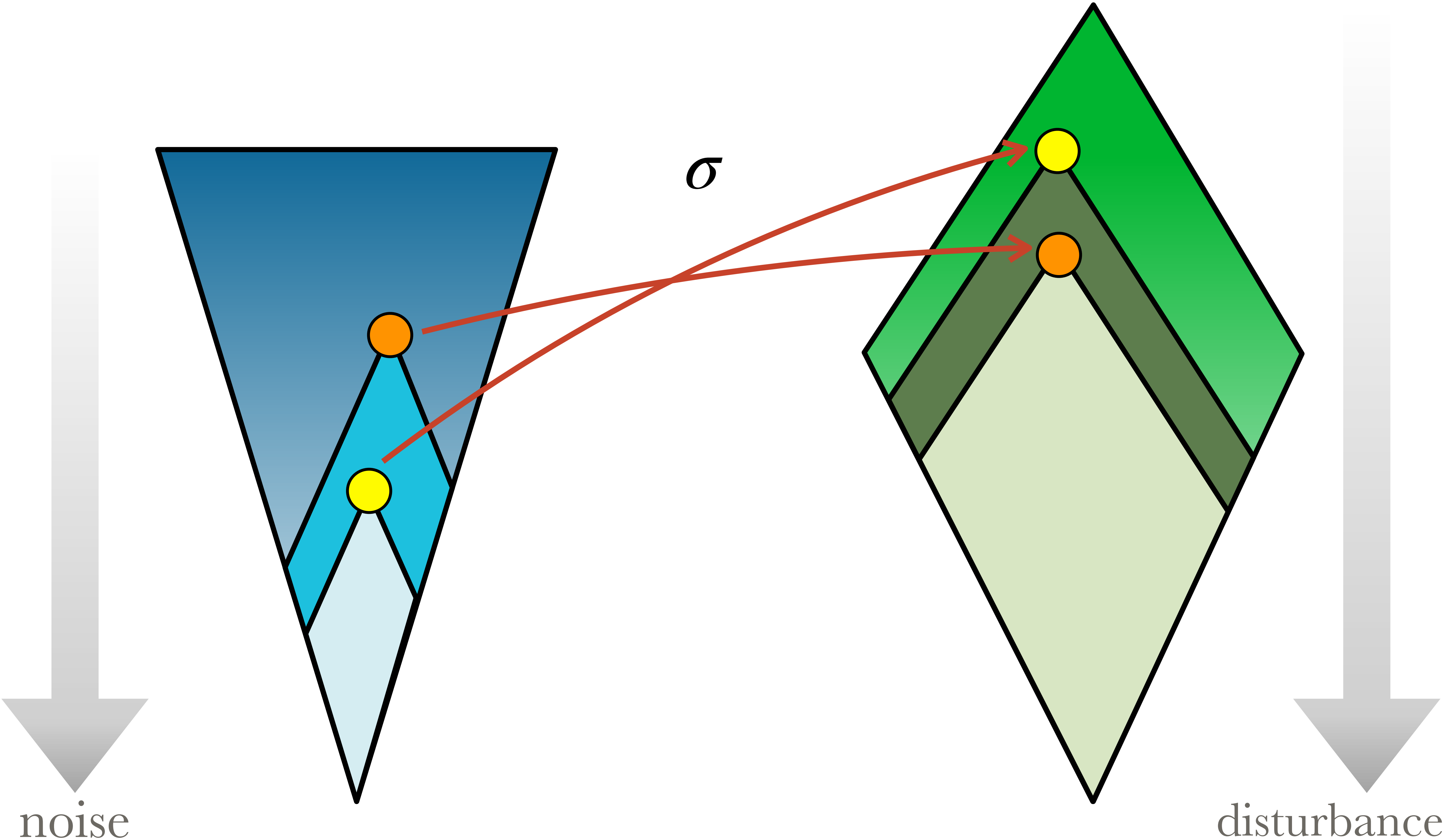


$$tr[\varrho A(x)] = tr[U\varrho \otimes \xi U^* \; I \otimes P_A(x)]$$

D) qualitative noise-disturbance relation



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Does **not** hold in all operational theories:

$$A \preccurlyeq B \quad \Leftarrow \quad \sigma(A) \supseteq \sigma(B)$$

quantifications of noise and disturbance

Basic requirements

$$A \precsim B \quad \Longrightarrow \quad \mathit{noise}(A) \leq \mathit{noise}(B)$$

$$\Lambda \precsim \Gamma \quad \Longrightarrow \quad \mathit{disturbance}(\Lambda) \leq \mathit{disturbance}(\Gamma)$$

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PRL **112,** 050401 (2014) PHYSICAL REVIEW LETTERS week ending 7 FEBRUARY 2014

Noise and Disturbance in Quantum Measurements: An Information-Theoretic Approach

Francesco Buscemi,[1,*] Michael J. W. Hall,[2,†] Masanao Ozawa,[3,‡] and Mark M. Wilde[4,§]

Theorem 1: For any measuring apparatus $\mathcal{M}$ and any nondegenerate observables X and Z, the following tradeoff between noise $\mathsf{N}(\mathcal{M}, X)$ and disturbance $\mathsf{D}(\mathcal{M}, Z)$ holds:

$$\mathsf{N}(\mathcal{M}, X) + \mathsf{D}(\mathcal{M}, Z) \geq -\log\, c, \qquad (1)$$

where $c := \max_{x,z} |\langle \psi^x | \varphi^z \rangle|^2$ and the log is in base 2.

closure maps

$c : 2^{\mathscr{A}} \to 2^{\mathscr{A}}$ is a closure map if

1. $c \circ c = c$
2. $X \subseteq c(X)$
3. $X' \subseteq X \;\Rightarrow\; c(X') \subseteq c(X)$

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Examples:

- topological closure
- linear span
- convex hull

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In a Galois connection $\tau\sigma$ and $\sigma\tau$ are closure maps.

Further, $\tau\sigma\tau = \tau$ and $\sigma\tau\sigma = \sigma$.

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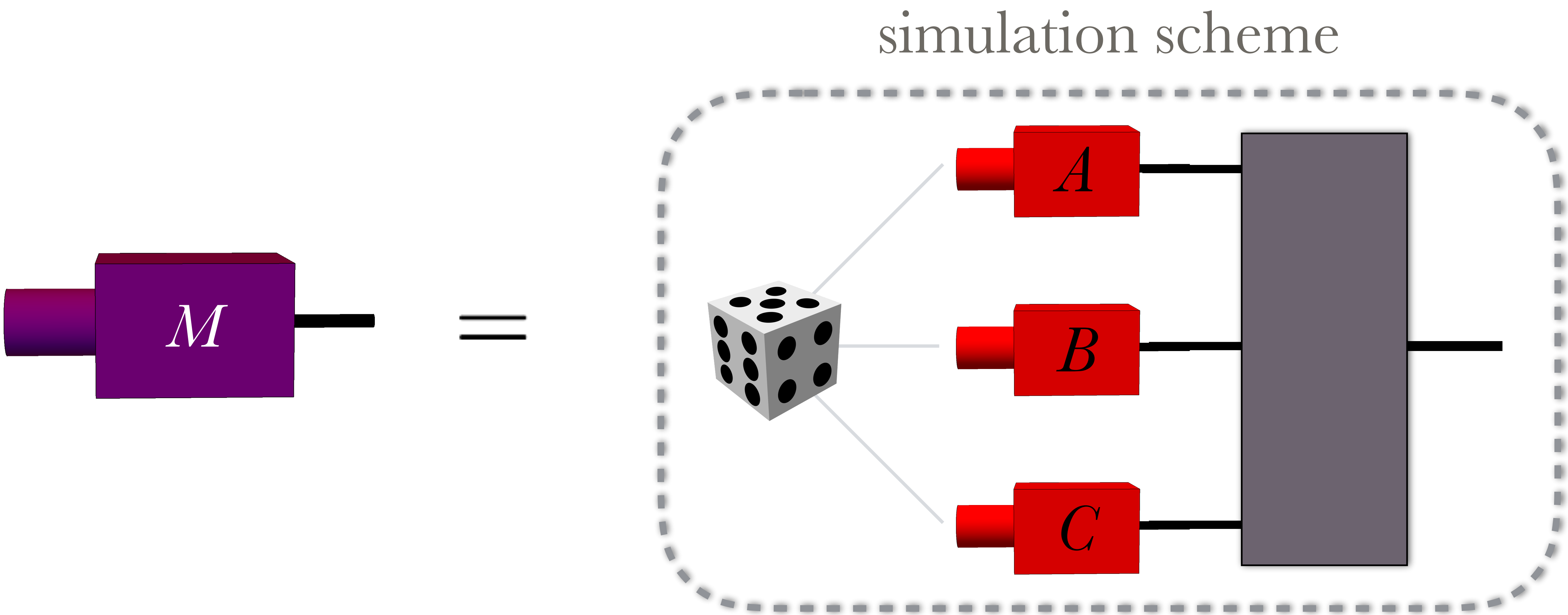
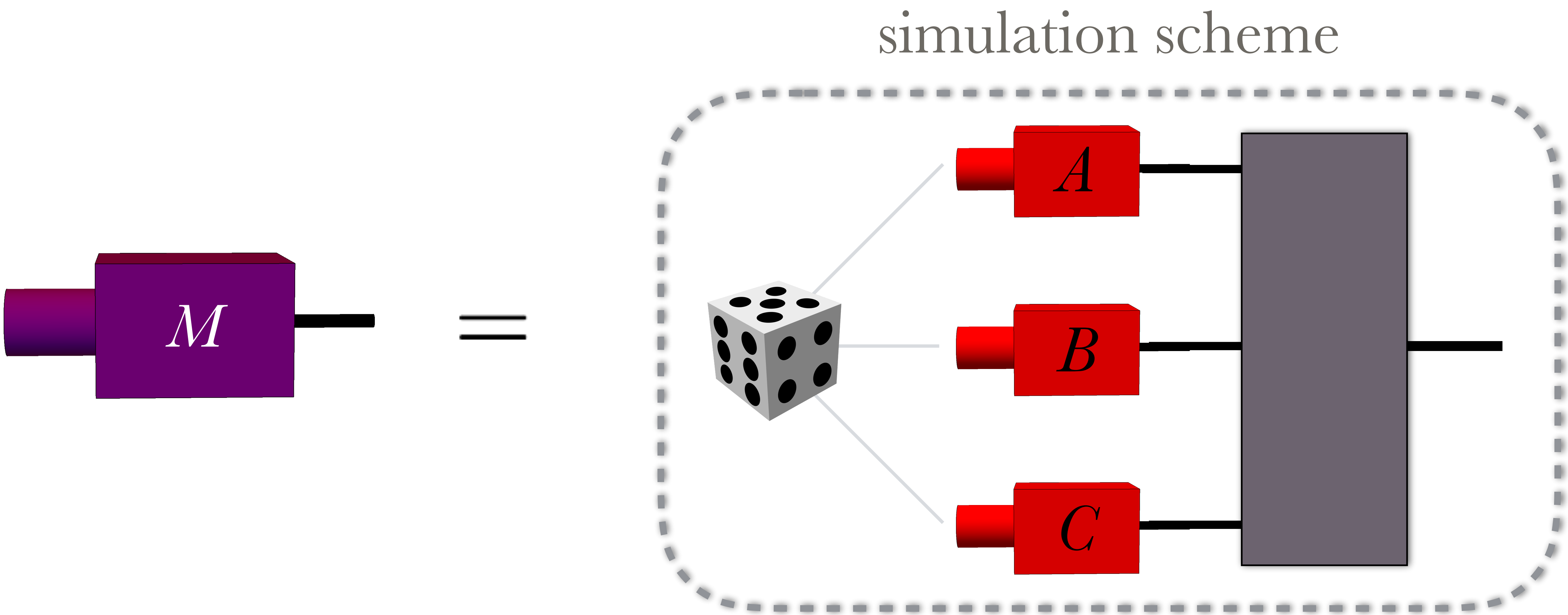
simulation closure



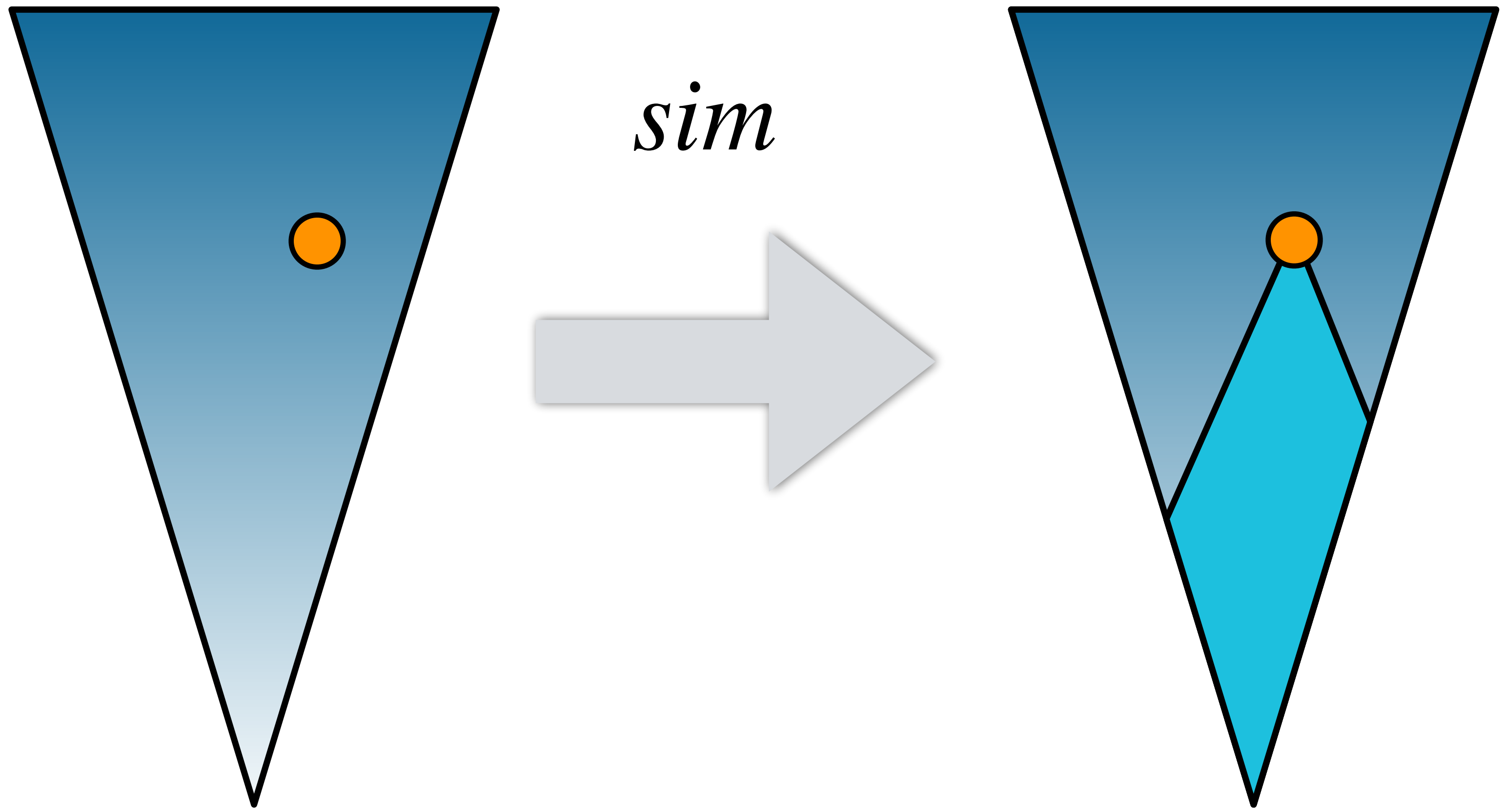
simulation closure



sim($\{A,B,C\}$) = {all observables that can be simulated from A,B and C with some randomization and post-processing}

simulation closure

$$sim(\{A\}) = \downarrow A$$



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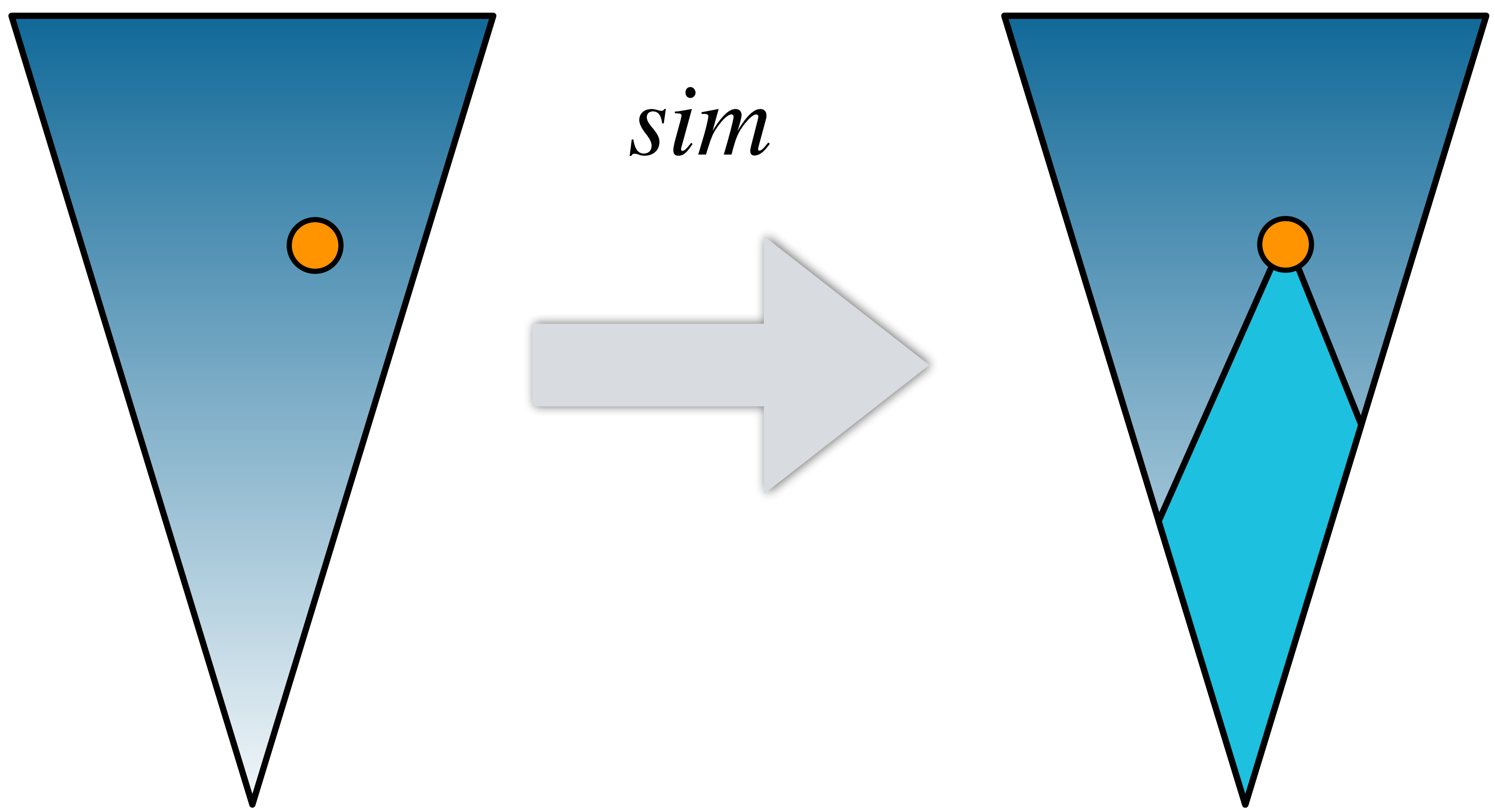


$$sim(\tau(Y)) = \tau(Y)$$

simulation closure

$$\tau\sigma(\{A\}) = sim(\{A\}) = \downarrow A$$

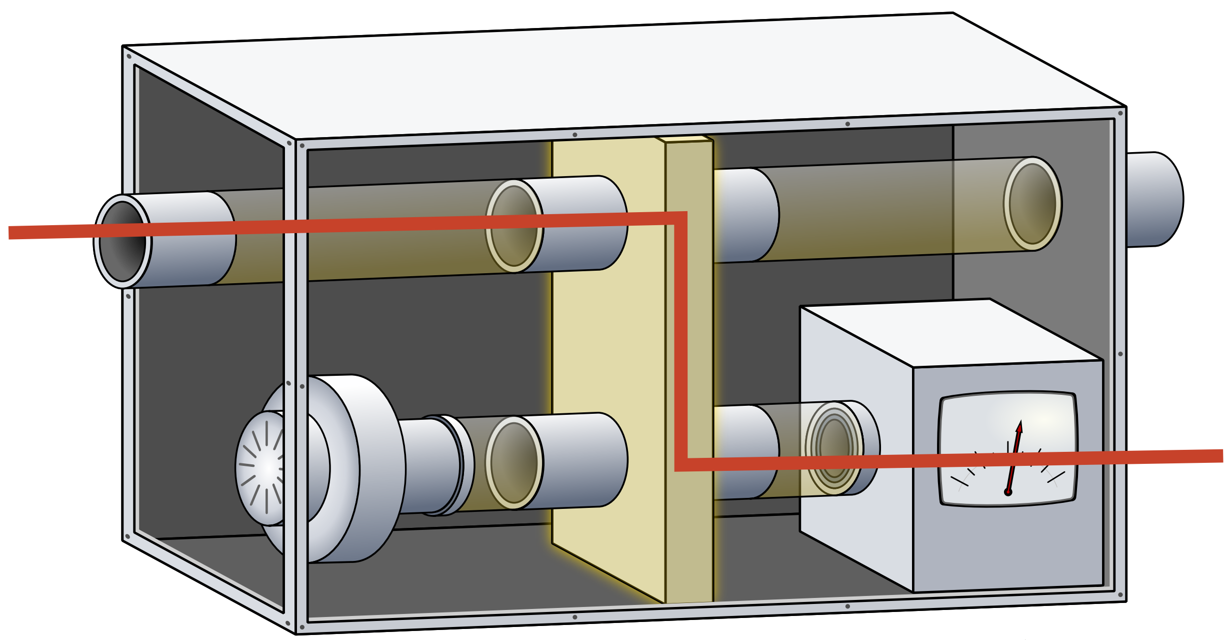
simulation closure

$$\tau\sigma(\{A\}) = sim(\{A\}) = \downarrow A$$

$$\tau\sigma(X) \supseteq sim(X)$$

physical interpretation of $\tau\sigma$

realization of A



A

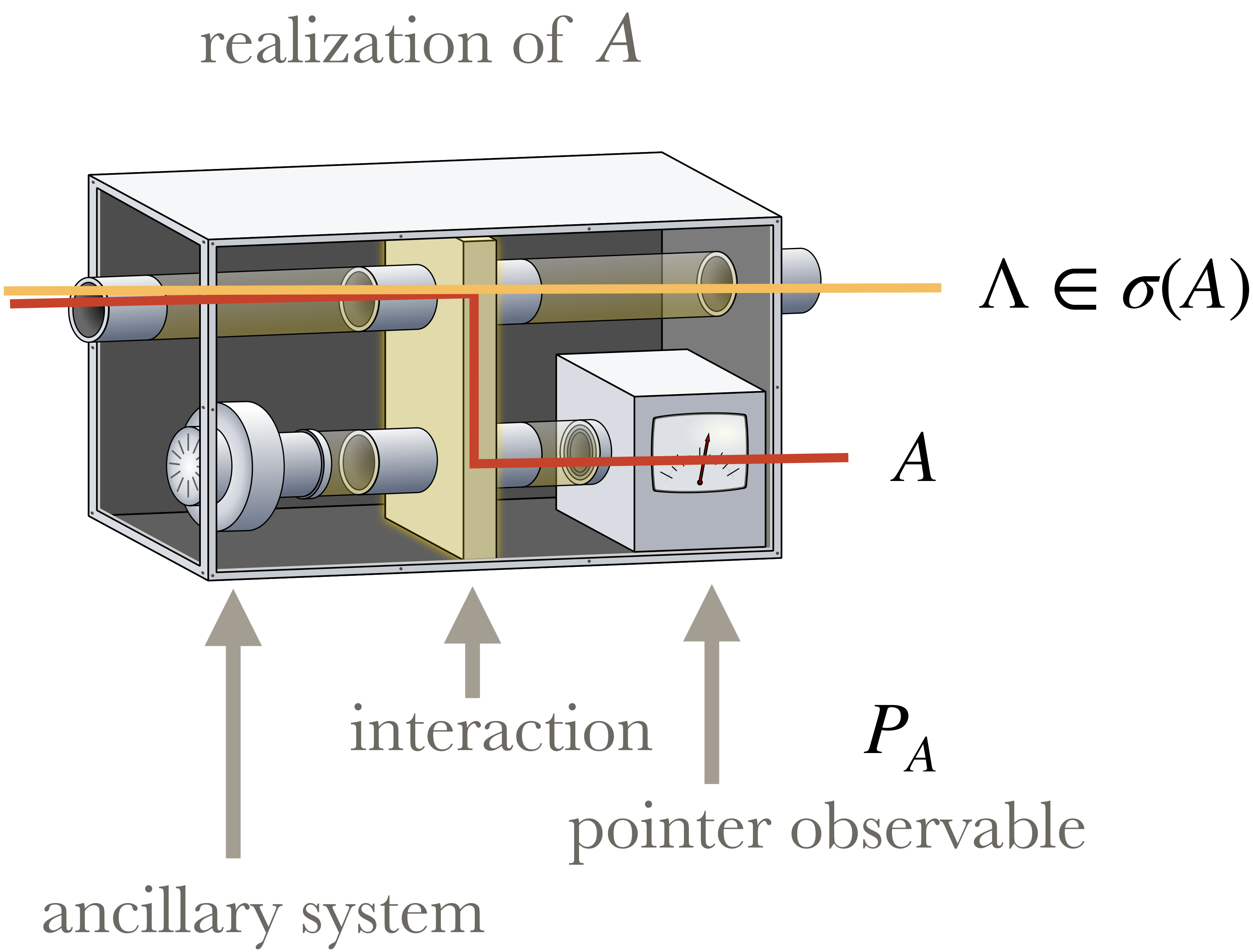
interaction

P_A

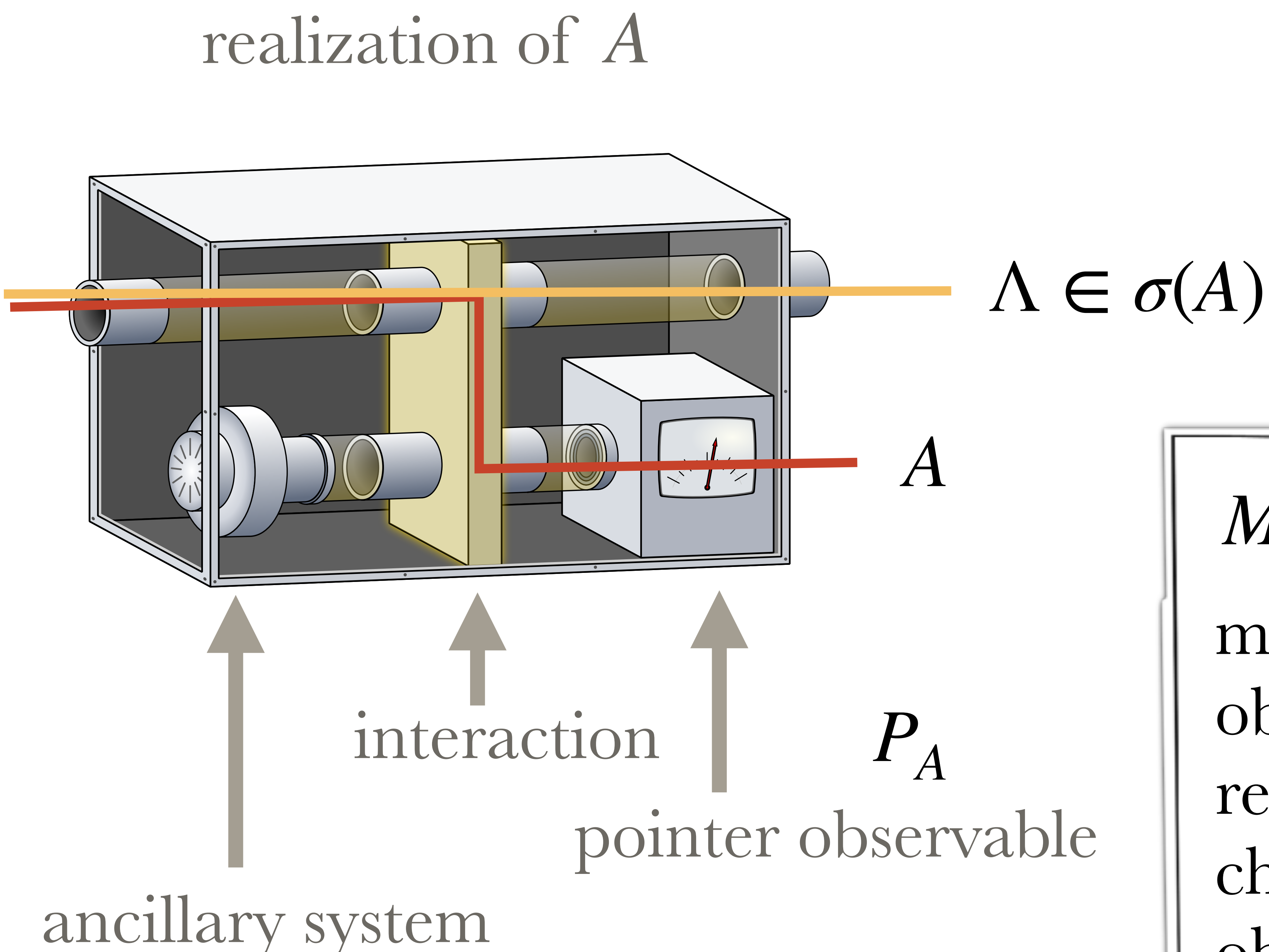
pointer observable

ancillary system

physical interpretation of $\tau\sigma$



physical interpretation of $\tau\sigma$

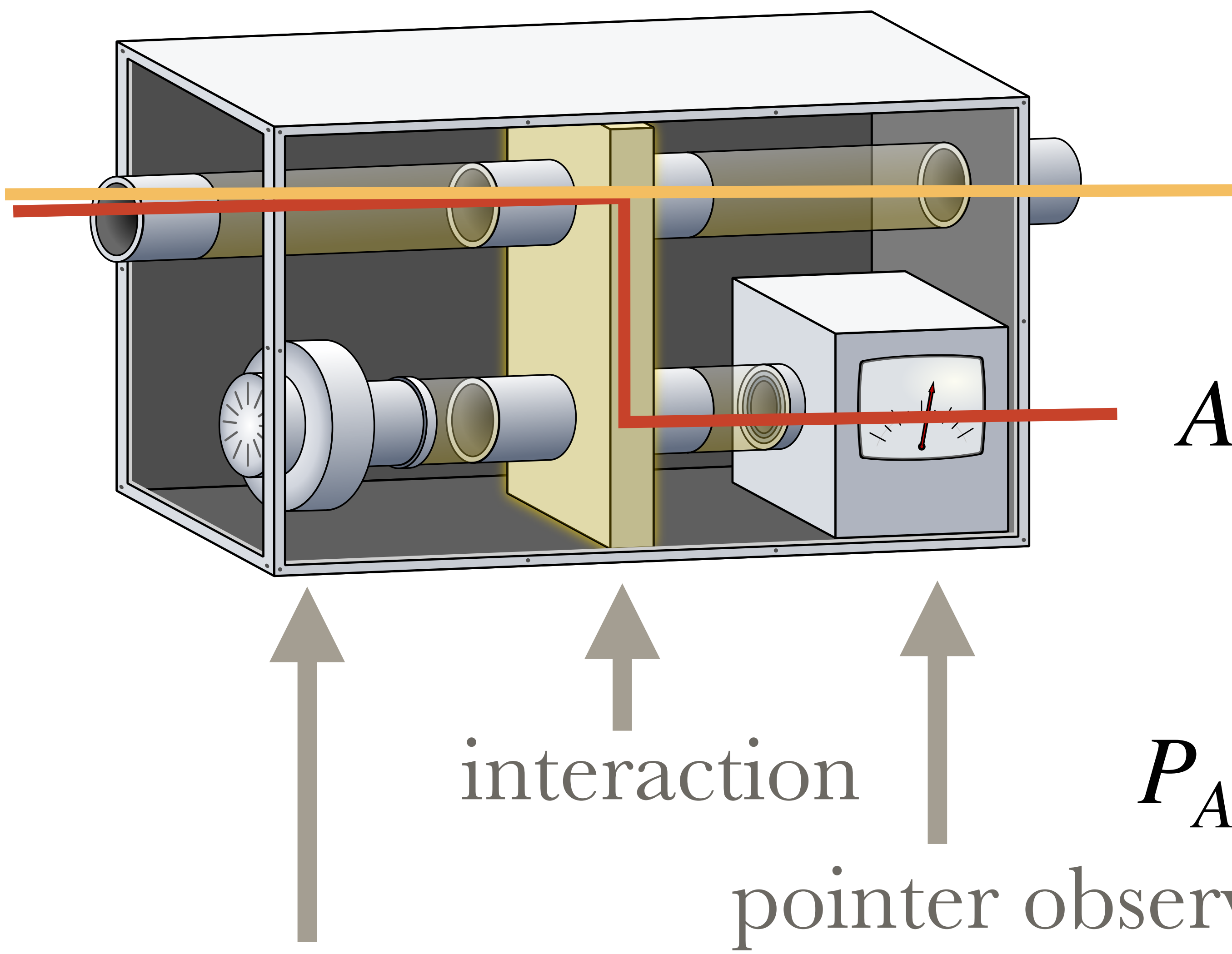


$M \in \tau\sigma(A)$

means that M can be obtained in *all* realizations of A by changing the pointer observable

physical interpretation of $\tau\sigma$

realization of $X = \{A, B\}$



$M \in \tau\sigma(X)$

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physical interpretation of $\tau\sigma$

$\tau\sigma(X)$ is the "information leak" required to implement X

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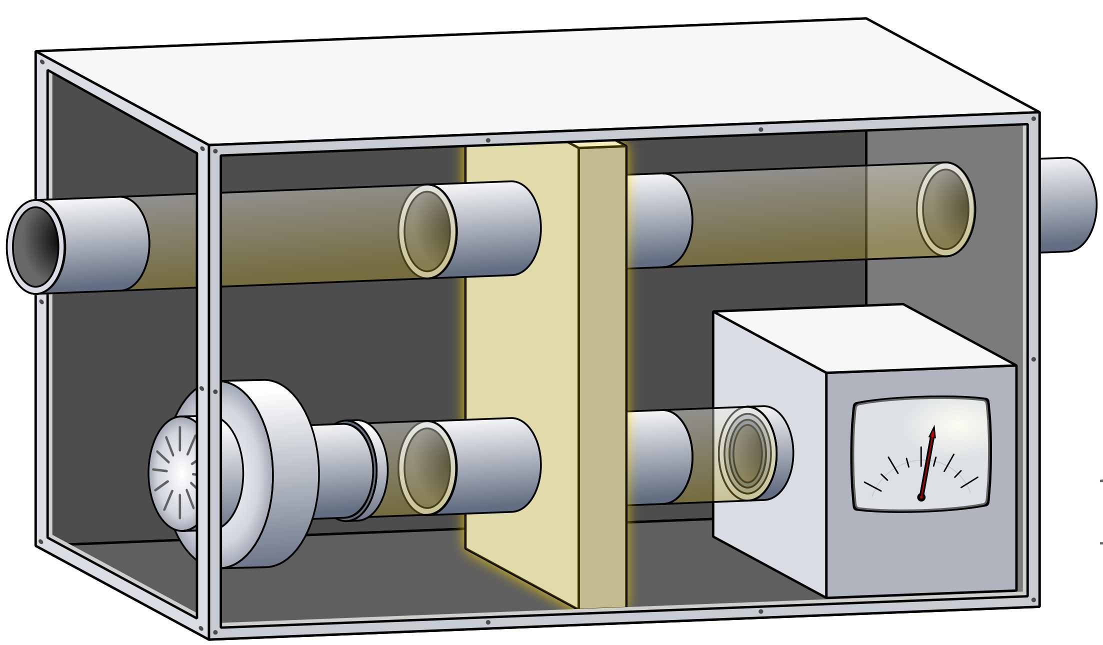
Example: it is possible that $\tau\sigma(\{A, B\}) = O$



$\sigma(\{A, B\}) = $ [discard-prepare]

pointer observables

$P_A = A, P_B = B$

interaction $U(\psi \otimes \varphi) = \varphi \otimes \psi$

physical interpretation of $\tau\sigma$

$\tau\sigma(X)$ is the "information leak" required to implement X

Example: it is possible that $\tau\sigma(\{A, B\}) = O$

However, $sim(X) \neq O$ for any countable subset X

physical interpretation of $\tau\sigma$

$\tau\sigma(X)$ is the "information leak" required to implement X

Example:

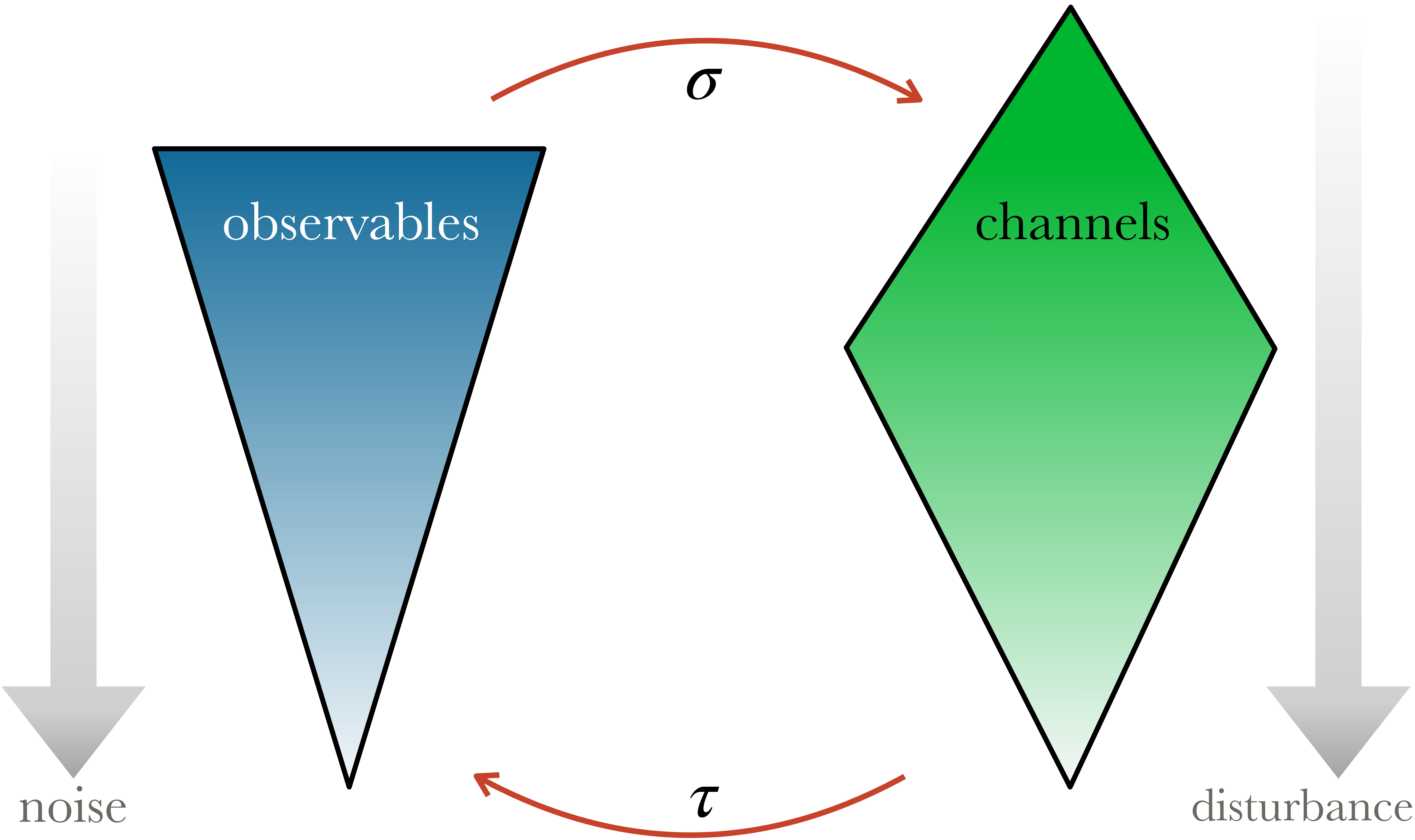
$$E(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E(2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

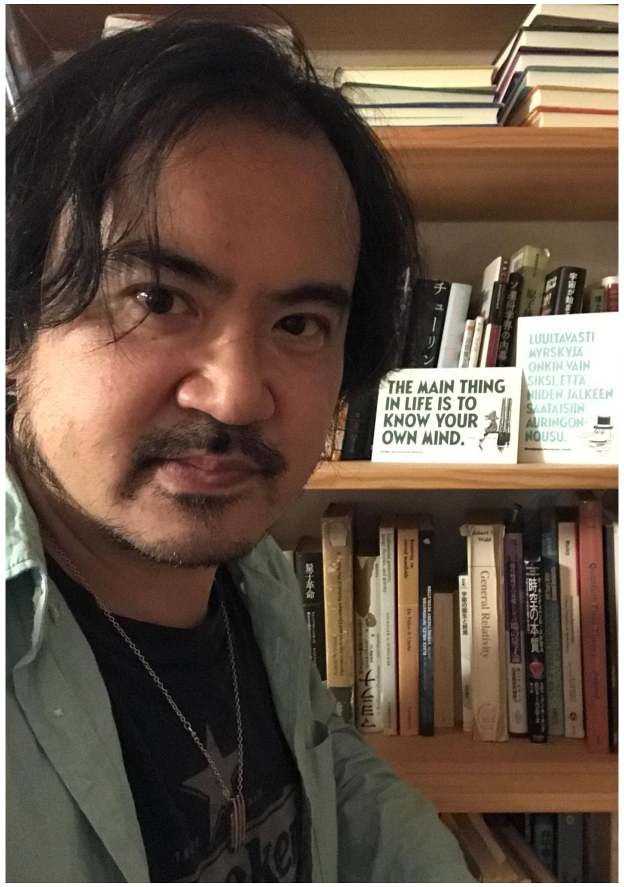
$$A(1) = E(1) + E(2) \qquad B(1) = E(1)$$

$$A(2) = E(3) \qquad B(2) = E(2) + E(3)$$

$$\tau\sigma(\{A, B\}) = \downarrow E \neq sim(\{A, B\})$$

Galois connection





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- TH and **Takayuki Miyadera** : Universality of sequential quantum measurements, 2015
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- TH, **Daniel Reitzner**, **Tomas Rybar** and **Mario Ziman** : Incompatibility of unbiased qubit observables and Pauli channels, 2018
- **Erkka Haapasalo**, TH and **Takayuki Miyadera** : The unavoidable information flow to environment in quantum measurements, 2018
- **Sergey Filippov**, TH and **Leevi Leppäjärvi** : Simulability of observables in general probabilistic theories, 2018

- **Claudio Carmeli**, TH, **Takayuki Miyadera, Alessandro Toigo** : Noise–Disturbance Relation and the Galois Connection of Quantum Measurements, 2019