Entangled Edge States of Corank 1 with Positive Partial Transposes

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Entangled mixed states

• Pure states are nonzero vectors $x \in \mathbb{C}^n$ up to constant. Geometrically, pure states form projective space $\mathbb{P}^n = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$. Pure states of two particle system are $\sum_i x_i \otimes y_i \in \mathbb{C}^n \otimes \mathbb{C}^m - \{0\}$.

• Separable pure states are vectors of the form $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^m$ in the quantum system of two particles. If we think of vectors in $\mathbb{C}^n \otimes \mathbb{C}^m$ as $m \times n$ matrices after choosing bases, the set of separable pure states is the set of rank 1 matrices. A non-separable nonzero vector in $\mathbb{C}^n \otimes \mathbb{C}^m$ is called an entangled pure states.

• In $\mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^m) = \mathbb{P}^{mn-1}$, the locus of separable states is the image of the Segre map

$$\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \longrightarrow \mathbb{P}^{mn-1}, \quad ([x],[y]) \mapsto [x \otimes y].$$

• Mixed states for the Hilbert space $\mathbb{C}^n$ are $n \times n$ positive (Hermitian and eigenvalues≥ 0) matrix of trace 1;

$$\rho = \sum_{i=1}^{k} p_i x_i x_i^\dagger \in M_n^+, \quad p_i \geq 0,\ x_i \in \mathbb{C}^n.$$

• Mixed states for two particle system are positive operators

$$\rho = \sum_{i=1}^{k} p_i v_i v_i^\dagger \in M_{mn}^+ = (M_n \otimes M_m)^+, \quad p_i \geq 0,\ v_i \in \mathbb{C}^n \otimes \mathbb{C}^m.$$

• A mixed state ρ is separable if we can choose v_i to be separable vectors $x_i \otimes y_i$ with $x_i \in \mathbb{C}^n$ and $y_i \in \mathbb{C}^m$.

PPT criterion [Choi 1982, Peres 1996]

If $\rho = \sum_{i=1}^{k} p_i (x_i \otimes y_i)(x_i \otimes y_i)^\dagger \in M_{mn} = M_n \otimes M_m$ with $p_i \geq 0$, then its partial transpose ρ^Γ is also positive.

$$\text{Separable} \Rightarrow \text{PPT}$$

Here the partial transpose of $\sum_i A_i \otimes B_i \in M_n \otimes M_m$ is $\sum_i A_i^t \otimes B_i$. The partial transpose of $(x_i \otimes y_i)(x_i \otimes y_i)^\dagger$ is $(\bar{x}_i \otimes y_i)(\bar{x}_i \otimes y_i)^\dagger \in M_{mn}^+$. In particular, if $\rho : \mathbb{C}^n \otimes \mathbb{C}^m \to \mathbb{C}^n \otimes \mathbb{C}^m$ is separable, there are nonzero separable vectors

$$x_i \otimes y_i \in \mathrm{Im}(\rho) \quad \text{such that} \quad \bar{x}_i \otimes y_i \in \mathrm{Im}(\rho^\Gamma).$$

The converse of the PPT criterion is not true in general.

- Which entangled PPT states violate the PPT criterion most?

A PPT entangled edge state of corank (k, ℓ) is $\rho \in M_{mn}$ s.t.

1. $\rho \geq 0$, $\text{corank}(\rho) = k$;

2. $\rho^\Gamma \geq 0$, $\text{corank}(\rho^\Gamma) = \ell$;

3. if $x \otimes y \in \text{Im}(\rho)$ and $\bar{x} \otimes y \in \text{Im}(\rho^\Gamma)$, then $x = 0$ or $y = 0$.

Problem. Find PPT entangled edge states.

Numerical constraints

[K-Kye-Lee, JMP 2011] If a PPT entangled edge state of corank (k, ℓ) exists, then either $k + \ell > m + n - 2$ or (*)

$$k + \ell = m + n - 2, \qquad \sum_{r+s=m-1} (-1)^r \binom{k}{r}\binom{\ell}{s} = 0.$$

The first nontrivial series of solutions is $m = n$, $k = 1$, $\ell = 2n - 3$.

[K-Kye-Na, CMP 2015] Extended to multipartite systems.

[Heo-K, LAA 2019] For many solutions (m, n, k, ℓ) to (*),
(Range criterion) $\exists$ $D, E \le \mathbb{C}^m \otimes \mathbb{C}^n$ of codim k, ℓ resp. such that

$$x \otimes y \in D,\ \bar{x} \otimes y \in E \Rightarrow x \otimes y = 0.$$

The proofs require basic algebraic geometry and topology.

$n \otimes n$ **PPT edge states of corank** $(1, 2n-3)$

[Kye-Osaka, JMP 2012] Constructed an explicit $3 \otimes 3$ PPT edge state of corank $(1, 3)$.

[Choi-K-Kye, 2019] Constructed $n \otimes n$ PPT states for $n \geq 3$, which are PPT edge states of corank $(1, 2n-3)$ for $3 \leq n \leq 1000$.

We conjecture that our construction gives us $n \otimes n$ PPT edge states of corank $(1, 2n-3)$ for all $n \geq 3$.

Ingredients

• We need positive matrices, easy to control: kernel, image, rank, many zeros, etc.

• We need bilinear equations.

$x \otimes y = \sum_{i,j=1}^{n} x_i y_j \, e_i \otimes e_j$ if $x = \sum_{i=1}^{n} x_i e_i$, $y = \sum_{i=1}^{n} y_i e_i$.

If $\mathrm{Ker}(\rho)$ is generated by $\sum_{i,j} a_{ij} \, e_i \otimes e_j$, then

$$x \otimes y \in \mathrm{Im}(\rho) = (\mathrm{Ker}(\rho))^{\perp} \quad \Leftrightarrow \quad \sum_{i,j} \bar{a}_{ij} x_i y_j = 0$$

which is a single bilinear equation in x_i, y_j.

Likewise, $\bar{x} \otimes y \in \mathrm{Im}(\rho^{\Gamma})$ is equivalent to $(2n-3)$ bilinear equations in $\bar{x}_i$ and y_j.

Positive matrices of corank 1

For $z \in \mathbb{C}$ with $|z| = 1$, we will use

$$P_2(z) = \begin{pmatrix} 1 & z \\ z^{-1} & 1 \end{pmatrix} \in M_2$$

which is positive of corank one with kernel spanned by $(1, -z^{-1})^t$.

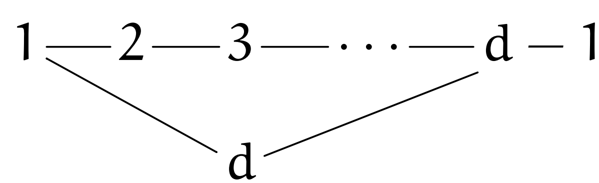
For $d \geq 4$ even and complex numbers $z_2, \cdots, z_d$ with $|z_j| = 1$, consider

$$P_d(z_2, \cdots, z_d) = \begin{pmatrix} 2 & z_2 & 0 & 0 & \cdots & z_d \\ z_2^{-1} & 2 & z_3 z_2^{-1} & 0 & \cdots & 0 \\ 0 & z_3^{-1} z_2 & 2 & z_4 z_3^{-1} & \cdots & 0 \\ 0 & 0 & z_4^{-1} z_3 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ z_d^{-1} & 0 & 0 & \cdots & z_d^{-1} z_{d-1} & 2 \end{pmatrix} \in M_d.$$

Note that $P_d(z_2, \cdots, z_d)$ is unitarily conjugate to

$$P_d(-1, 1, \cdots, -1) = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

which is the Cartan matrix $(2\delta_{ij} - a_{ij})$ of the graph with d vertices and d edges that form a cycle



where (a_{ij}) is the adjacency matrix defined by $a_{ij} = 1$ if the vertices i and j are connected by an edge and $a_{ij} = 0$ if not.

Bilinear equations

Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in \mathbb{C}^n$.
For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{C}^*)^n$, let (twisted Plucker coordinates)

$$[i, j]_\alpha = x_i y_j - \alpha_i^{-1} \alpha_j x_j y_i,$$

When $\alpha_i = 1$ for all i, $(2n-3)$ general linear combinations of $[i, j]$ are zero if and only if x and y are parallel. (For algebraic geometers: $\dim \mathrm{Gr}(2, n) = 2n - 4 < 2n - 3$.)

If we further insist the orthogonality equation $x \cdot y = \sum_i \bar{x}_i y_i = 0$, the only solution is $x = 0$ or $y = 0$.

The same hold true if we keep α_i sufficiently close to 1.

In our construction of PPT states, we cannot keep all of the above equations and will have to give up the orthogonality equation.

Explicit bilinear equations

For $\alpha, \beta \in (\mathbb{C}^*)^n$ with $\alpha_i^{-1}\alpha_j \neq \beta_i^{-1}\beta_j$, we can solve the system

$$[1,k]_\alpha - [2,k-1]_\alpha + [3,k-2]_\alpha - \cdots + (-1)^{\lfloor \frac{k}{2} \rfloor - 1}[\lfloor \frac{k}{2} \rfloor, k+1-\lfloor \frac{k}{2} \rfloor]_\alpha = 0,$$

$$[n-\ell, n]_\beta - [n-\ell+1, n-1]_\beta + \cdots + (-1)^{\lfloor \frac{\ell-1}{2} \rfloor}[n-\ell+\lfloor \frac{\ell-1}{2} \rfloor, n-\lfloor \frac{\ell-1}{2} \rfloor]_\beta = 0$$

of $2n-3$ equations for $k = 2, 3, \ldots, n$ and $\ell = 1, 2, \ldots, n-2$, to find that the solutions (x_i, y_j) have many zeros and nonzero entries are parallel after twists.

$3 \otimes 3$ **PPT edge state of corank 1**

Let $(\alpha_1, \alpha_2, \alpha_3) = (1, \alpha, \alpha^2)$ with $|\alpha| = 1$ and $(\beta_1, \beta_2, \beta_3) = (1, 1, 1)$. We write $\alpha_{i,j} = \alpha_i^{-1}\alpha_j$ and $\beta_{i,j} = \beta_i^{-1}\beta_j$. We also assume $\alpha_{i,j} \neq \beta_{i,j}$ for $1 \leq i < j \leq 3$.

Let ρ^Γ be the 9×9 matrix defined as follows:

(i) The (e_{12}, e_{21})-principal submatrix is $P_2(\alpha_{1,2})$.

(ii) The (e_{13}, e_{31})-principal submatrix is $2P_2(\alpha_{1,3})$.

(iii) The (e_{23}, e_{32})-principal submatrix is $P_2(\beta_{2,3})$.

(iv) The (e_{11}, e_{22}, e_{33})-principal submatrix is rI_3 for $r > 1$ to be determined later.

(v) All the other entries are zero.

$$\rho^{\Gamma} = \left(\begin{array}{ccc|ccc|ccc} r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \alpha_{1,2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 2\alpha_{1,3} & \cdot & \cdot \\ \hline \cdot & \alpha_{1,2}^{-1} & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \beta_{2,3} & \cdot \\ \hline \cdot & \cdot & 2\alpha_{1,3}^{-1} & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2,3}^{-1} & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r \end{array}\right)$$

$\bar{x} \otimes y \in \mathrm{Im}\rho^{\Gamma}$ if and only if

$$\bar{x}_1 y_2 = \alpha \bar{x}_2 y_1, \quad \bar{x}_1 y_3 = \alpha^2 \bar{x}_3 y_1, \quad \bar{x}_2 y_3 = \bar{x}_3 y_2.$$

$$\rho = \left(\begin{array}{ccc|ccc|ccc} r & \cdot & \cdot & \cdot & \alpha_{1,2}^{-1} & \cdot & \cdot & \cdot & 2\alpha_{1,3}^{-1} \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{1,2} & \cdot & \cdot & \cdot & r & \cdot & \cdot & \cdot & \beta_{2,3}^{-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 2\alpha_{1,3} & \cdot & \cdot & \cdot & \beta_{2,3} & \cdot & \cdot & \cdot & r \end{array}\right)$$

ρ is positive of corank 1 $\Leftrightarrow$ r is the largest root $\hat{r}$ of $\det \rho = 0$.
$x \otimes y \in \mathrm{Im}\rho$ if and only if

$$(-2\alpha^2 + \hat{r}^{-1}\alpha)x_1y_1 + (2\alpha\hat{r}^{-1} - 1)x_2y_2 + (\hat{r} - \hat{r}^{-1})x_3y_3 = 0.$$

It is easy to see that ρ is a PPT edge state.

$4 \otimes 4$ **edge states of corank one**

Let $\alpha_i, \beta_i \in \mathbb{C}$ with $|\alpha_i| = |\beta_i| = 1$ for $1 \le i \le 4$.
Let ρ^Γ be the 16×16 matrix defined as follows:

(i) The (e_{12}, e_{21})-principal submatrix is $P_2(\alpha_{1,2})$.
(ii) The (e_{13}, e_{31})-principal submatrix is $2P_2(\alpha_{1,3})$.
(iii) The $(e_{14}, e_{23}, e_{32}, e_{41})$-principal submatrix is $P_4(1, \alpha_{2,3}, \alpha_{1,4})$.
(iv) The (e_{24}, e_{42})-principal submatrix is $2P_2(\beta_{2,4})$.
(v) The (e_{34}, e_{43})-principal submatrix is $P_2(\beta_{3,4})$.
(vi) The $(e_{11}, e_{22}, e_{33}, e_{44})$-principal submatrix is rI_4 for $r > 1$ to be determined later.
(vii) All the other entries are zero.

$$
\left(\begin{array}{cccc|cccc|cccc|cccc}
r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \alpha_{1,2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 2\alpha_{1,3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 2 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{1,4} & \cdot & \cdot & \cdot \\
\hline
\cdot & \alpha_{1,2}^{-1} & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 2 & \cdot & \cdot & \alpha_{2,3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & 2\beta_{2,4} & \cdot & \cdot \\
\hline
\cdot & \cdot & 2\alpha_{1,3}^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{2,3}^{-1} & \cdot & \cdot & 2 & \cdot & \cdot & \alpha_{2,3}^{-1}\alpha_{1,4} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \beta_{3,4} & \cdot \\
\hline
\cdot & \cdot & \cdot & \alpha_{1,4}^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{2,3}\alpha_{1,4}^{-1} & \cdot & \cdot & 2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2\beta_{2,4}^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{3,4}^{-1} & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r
\end{array}\right)
$$

$$
\rho = \left(\begin{array}{cccc|cccc|cccc|cccc}
r & \cdot & \cdot & \cdot & \cdot & \alpha_{1,2}^{-1} & \cdot & \cdot & \cdot & \cdot & 2\alpha_{1,3}^{-1} & \cdot & \cdot & \cdot & \cdot & \alpha_{1,4}^{-1} \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha_{1,2} & \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot & \cdot & \cdot & \alpha_{2,3}^{-1} & \cdot & \cdot & \cdot & \cdot & 2\beta_{2,4}^{-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \alpha_{2,3}\alpha_{1,4}^{-1} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2\alpha_{1,3} & \cdot & \cdot & \cdot & \cdot & \alpha_{2,3} & \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot & \cdot & \cdot & \beta_{3,4}^{-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{2,3}^{-1}\alpha_{1,4} & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\alpha_{1,4} & \cdot & \cdot & \cdot & \cdot & 2\beta_{2,4} & \cdot & \cdot & \cdot & \cdot & \beta_{3,4} & \cdot & \cdot & \cdot & \cdot & r
\end{array}\right).
$$

This is a PPT edge state of corank 1 for suitable α_i, β_j.

It is straightforward to generalize the above to $n \otimes n$ PPT states for $n \geq 3$. The condition (*)

$$x \otimes y \in \mathrm{Im}(\rho), \quad \bar{x} \otimes y \in \mathrm{Im}(\rho^\Gamma) \quad \Rightarrow \quad x \otimes y = 0$$

can be checked by Mathematica.

Theorem. For $3 \leq n \leq 1000$, the above construction gives us $n \otimes n$ PPT edge states of corank 1.

It took 8 hours with an ordinary laptop computer to check the condition (*) for $n \leq 1000$. If we run longer, we will get more PPT edge states.

Conjecture. The above construction gives us $n \otimes n$ PPT edge states of corank 1 for all $n \geq 3$.

Thank you for your attention.

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